M208

Pure mathematics

Book C

Linear algebra

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Unit C1

Linear equations and matrices

Introduction to Book C

Systems of linear equations in several variables arise in areas as diverse as science, technology and economics. The solutions to such systems can provide answers to a wide range of problems, from supply and demand dependencies in economics to working out currents in electrical networks. Apart from such practical applications, solving systems of linear equations is also an interesting mathematical problem in itself. Much effort has been devoted to solving systems of linear equations. You will see that this process is not always straightforward, especially if the number of variables is large.

One key issue is whether a given system of linear equations has any solutions at all. As a specific example of the types of situation that may occur, consider the following three rather similar pairs of linear equations in the variables x and y.

$$\begin{aligned} x + 3y &= 5 \\ -2x + 6y &= 2 \end{aligned} \qquad \begin{aligned} -x + 3y &= 5 \\ -2x + 6y &= 2 \end{aligned} \qquad \begin{aligned} -x + 3y &= 1 \\ -2x + 6y &= 2 \end{aligned}$$

The first pair of equations has the unique solution $x = 2$, $y = 1$, whereas the second pair has no solutions, and the third pair has infinitely many solutions (for example, $x = -1$, $y = 0$, and $x = 0$, $y = \frac{1}{3}$). You will see that these different outcomes may be understood:

- algebraically, by studying the *matrix* of coefficients of the equations, and introducing a function of these coefficients, called the *determinant*
- geometrically, by interpreting solutions of the equations as points of intersection of the corresponding pairs of straight lines drawn in an (x, y)-plane.

The algebraic approach as well as the geometric approach can be generalised for systems of linear equations that involve more than two variables. The geometric approach will require us to use a generalisation of the plane called *n-dimensional Euclidean space*, whose elements are of the form $(x_1, x_2, \dots, x_n)$, where $x_1, x_2, \dots, x_n$ are real numbers. Depending on the context, we will interpret these elements as either *points* or *vectors* in Euclidean space. Although it is only easy to visualise objects in n-dimensional space when $n = 1$, 2 or 3, this more general Euclidean space is a convenient environment in which to develop the theory needed to analyse the solutions of systems of linear equations.

You will see that a key tool in this theory is the concept of a *linear transformation* which, in its basic form, is a function from one Euclidean space to another that preserves certain aspects of the geometric structure of the Euclidean space. For example, the function

$$t(x, y) = (x + 3y, -2x + 6y)$$

is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, which is closely related to the first pair of equations above. Indeed, solving that pair of equations is equivalent to finding a point (x, y) in $\mathbb{R}^2$ such that the function t maps

(x, y) to the point $(5, 2)$. This suggests that we can obtain information about the solutions of systems of linear equations by studying the corresponding linear transformations.

But linear transformations arise in situations apart from that of solving equations. For example, they are needed to manipulate computer graphic images, as illustrated in Figure 1.

Figure 1 The effect of a rotation, reflection and shear on an image

Finally, the range of available linear transformations can be increased greatly by introducing the notion of a *vector space*. This is a generalisation of n-dimensional Euclidean space, and it may be finite-dimensional or infinite-dimensional. The elements of a vector space are sometimes called *vectors*, but they can be very general objects; for example, you will look at vector spaces whose elements are real functions, and linear transformations between such vector spaces that arise from operations on real functions such as differentiation and integration. In this way, vector spaces and their associated linear transformations form a very general context in which many seemingly unrelated problems can be studied using similar techniques.

In this book on linear algebra you will learn about all these concepts: solving systems of linear equations, matrices, vector spaces and linear transformations. You will also use this theory to classify conics and quadrics.

Introduction

In this first unit of linear algebra you will begin by considering systems of linear equations in two and three unknowns. You will then see how matrices can be used as a concise way of representing systems of linear equations, before going on to study matrices themselves. You will see how properties of the matrix of coefficients may be used to quickly determine whether the system of linear equations has a unique solution.

Many of the ideas and methods you will meet in this unit will also be used in the subsequent three units on linear algebra.

1 Systems of linear equations

In this section you will revise systems of linear equations in two and three unknowns and see how these ideas extend to systems in more unknowns.

Recall that a **system of linear equations** in two (or three) unknowns is a collection of linear equations each written in terms of a set of two (or three) unknowns. A **solution** to a system of linear equations is an assignment of values to the unknowns that makes all the equations hold simultaneously; therefore such a system is also called a **system of simultaneous linear equations** in the given set of unknowns.

1.1 Systems in two and three unknowns

Systems in two unknowns: one equation

In Unit A1 *Sets, functions and vectors*, you saw that an equation of the form

$$ax + by = c$$

where a, b and c are real numbers, and a and b are not both zero, represents a line in $\mathbb{R}^2$. There are infinitely many solutions to this equation – one corresponding to each point on the line.

Systems in two unknowns: two equations

The solutions to the following system of two linear equations

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

in the two unknowns x and y, where $a, b, \ldots, f$ are real numbers, correspond to the points of intersection of these two lines in $\mathbb{R}^2$.

Now, two arbitrary lines in $\mathbb{R}^2$ may intersect at a unique point, be parallel, or coincide, which means that solving a system of two linear equations in two unknowns yields exactly one of the following three situations.

- There is a unique solution, when the two lines represented by the equations intersect at a unique point, as illustrated in Figure 2.

  For example, the system

  $$\begin{aligned} x - y &= -1 \\ 2x + y &= 4 \end{aligned}$$

  has the unique solution $x = 1$, $y = 2$, corresponding to the unique point of intersection $(1, 2)$ of the two lines in $\mathbb{R}^2$.

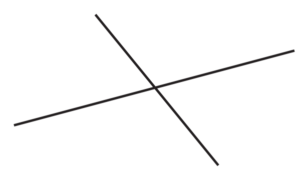


Figure 2 Two lines intersecting at a unique point

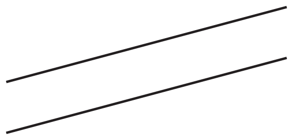


Figure 3 Two parallel lines with no point of intersection

- There is no solution, when the two lines represented by the equations are parallel, as illustrated in Figure 3.

  For example, the system

  $$\begin{aligned} x - y &= -1 \\ x - y &= 1 \end{aligned}$$

  represents two parallel lines in $\mathbb{R}^2$ that do not intersect, and so the system has no solution.

- There are infinitely many solutions, when the two lines represented by the equations coincide, as illustrated in Figure 4.

  For example, the system

  $$\begin{aligned} -6x + 3y &= -6 \\ 2x - y &= 2 \end{aligned}$$

  has infinitely many solutions, as the two equations represent the same line in $\mathbb{R}^2$: the equations are a multiple of one another. In a sense, the two lines intersect at all of their points; that is, each pair of values for x and y satisfying $2x - y = 2$ is a solution to this system.

Figure 4 Two coincident lines

Systems in three unknowns: one equation

In Unit A1 you saw that an equation of the form

$$ax + by + cz = d$$

where a, b, c and d are real numbers, and a, b and c are not all zero, represents a plane in $\mathbb{R}^3$. There are infinitely many solutions to this equation – one corresponding to each point in the plane.

Systems in three unknowns: two equations

The solutions to the system of two linear equations

$$\begin{aligned} ax + by + cz &= d \\ ex + fy + gz &= h \end{aligned}$$

in the three unknowns x, y and z, where $a, b, \ldots, h$ are real numbers, correspond to the points of intersection of these two planes in $\mathbb{R}^3$.

Two arbitrary planes in $\mathbb{R}^3$ may intersect, be parallel or coincide. In general, when two distinct planes in $\mathbb{R}^3$ intersect, the set of common points is a line that lies in both planes. This means that solving a system of two linear equations in three unknowns yields exactly one of the following two situations.

- There is no solution, when the two planes represented by the equations are parallel, as illustrated in Figure 5.

  For example, the system

  $$\begin{aligned} x + y + z &= 1 \\ x + y + z &= 2 \end{aligned}$$

  represents two parallel planes in $\mathbb{R}^3$ and so has no solutions.

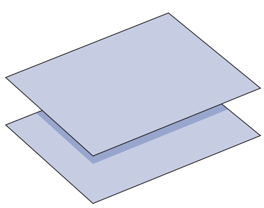


Figure 5 Two parallel planes

- There are infinitely many solutions, when the two planes represented by the equations coincide, or when they intersect in a line, as illustrated in Figures 6 and 7, respectively.

  For example,

  $$\begin{aligned} x + y + z &= 1 \\ 2x + 2y + 2z &= 2 \end{aligned}$$

  has infinitely many solutions, as the two equations represent the same plane in $\mathbb{R}^3$. Each set of values for x, y and z satisfying $x + y + z = 1$ is a solution to this system, such as $x = 1$, $y = 0$, $z = 0$ and $x = -2$, $y = 4$, $z = -1$.

  Similarly, the system

  $$\begin{aligned} x + y + z &= 1 \\ x + y \phantom{{}+z} &= 1 \end{aligned}$$

  has infinitely many solutions: the planes in $\mathbb{R}^3$ represented by the two equations intersect in a line. The z-coordinate of each point on this line is zero, and so the line lies in the (x, y)-plane. Each set of values for x, y and z satisfying $x + y = 1$ and $z = 0$ is a solution to this system, such as $x = 1$, $y = 0$, $z = 0$ and $x = 5$, $y = -4$, $z = 0$.

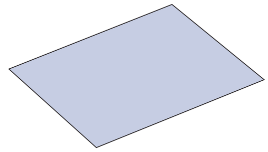


Figure 6 Two coincident planes

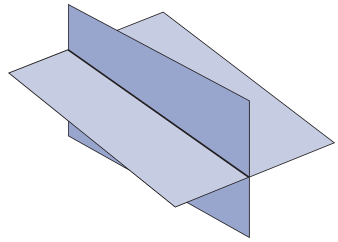


Figure 7 Two planes intersecting in a line

Systems in three unknowns: three equations

In a similar way, the solutions to the system of three linear equations

$$\begin{aligned} ax + by + cz &= d \\ ex + fy + gz &= h \\ ix + jy + kz &= l \end{aligned}$$

in the three unknowns x, y and z, where $a, b, \ldots, l$ are real numbers, correspond to the points of intersection of these three planes in $\mathbb{R}^3$.

Three arbitrary planes in $\mathbb{R}^3$ may meet each other in a number of different ways. We illustrate these possibilities below. A system of three linear equations in three unknowns yields exactly one of the following three situations.

- There is a unique solution, when the three planes represented by the equations intersect at a unique point, as illustrated in Figure 8.

  For example, the system

  $$\begin{aligned} x + y + z &= 1 \\ x + y \phantom{{}+z} &= 1 \\ x \phantom{{}+y} - z &= 0 \end{aligned}$$

  has the unique solution $x = 0$, $y = 1$, $z = 0$, corresponding to the unique point of intersection $(0, 1, 0)$ of the three planes in $\mathbb{R}^3$.

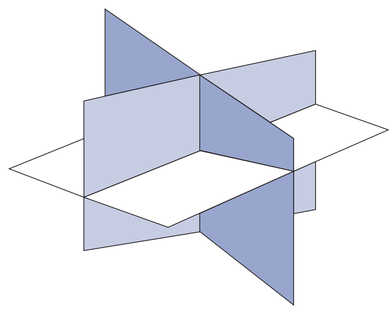


Figure 8 Three planes intersecting in a unique point

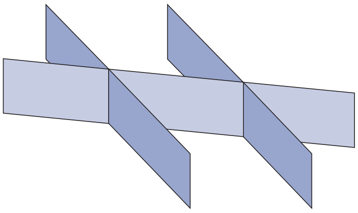


Figure 9 Three planes, two of which are parallel

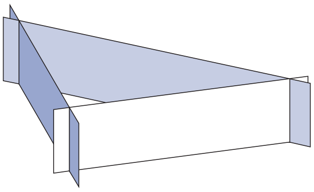


Figure 10 Three planes intersecting in pairs forming a prism

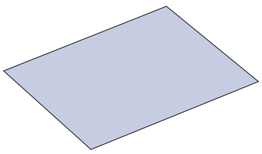


Figure 11 Three coincident planes

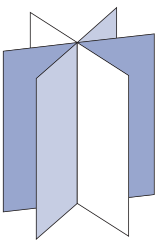


Figure 12 Three planes intersecting in a line

- There is no solution, when two (or three) of the planes represented by the equations are parallel, or when the three planes form a triangular prism, as illustrated in Figures 9 and 10, respectively.

  For example, the system

$$\begin{aligned} x + y + z &= 1 \\ x + y + z &= 2 \\ x + y - z &= 0 \end{aligned}$$

  represents three planes in $\mathbb{R}^3$, the first two of which are parallel, and so the system has no solutions.

  Similarly, the system

$$\begin{aligned} x + y \phantom{{}+z} &= 1 \\ x \phantom{{}+y} + z &= 1 \\ - y + z &= 1 \end{aligned}$$

  has no solutions: the planes in $\mathbb{R}^3$ represented by the three equations intersect in pairs, forming a triangular prism, and so there are no points common to *all three* planes.

- There are infinitely many solutions, when the three planes that the equations represent intersect either in a plane or in a line, as illustrated in Figures 11 and 12, respectively.

  For example, the system

$$\begin{aligned} x + y + z &= 1 \\ -x - y - z &= -1 \\ 2x + 2y + 2z &= 2 \end{aligned}$$

  has infinitely many solutions, as the three equations all represent the same plane in $\mathbb{R}^3$: the equations are multiples of one another. Each set of values for x, y and z satisfying $x + y + z = 1$ is a solution to this system, such as $x = 1$, $y = 0$, $z = 0$ and $x = -1$, $y = 3$, $z = -1$.

  Similarly, the system

$$\begin{aligned} x + y + z &= 1 \\ x + y \phantom{{}+z} &= 1 \\ x + y - z &= 1 \end{aligned}$$

  has infinitely many solutions: the planes in $\mathbb{R}^3$ represented by the three equations intersect in a line. The z-coordinate of each point on this line is zero, and so the line lies in the (x, y)-plane. Each set of values for x, y and z satisfying $x + y = 1$ and $z = 0$ is a solution to this system, such as $x = 1$, $y = 0$, $z = 0$ and $x = -5$, $y = 6$, $z = 0$.

1.2 Systems in n unknowns

The equations for a line in $\mathbb{R}^2$ and a plane in $\mathbb{R}^3$ are *linear equations* in two and three unknowns, respectively. Similarly, an equation of the form

$$ax + by + cz + dw = e$$

is a linear equation in the four unknowns x, y, z and w, where $a, \ldots, e$ are real numbers, and a, b, c and d are not all zero. In general, we can define a linear equation in any number of unknowns.

Definitions

An equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n, b$ are real numbers, and $a_1, \ldots, a_n$ are not all zero, is a **linear equation** in the n **unknowns** $x_1, x_2, \ldots, x_n$. The numbers a_i are the **coefficients**, and b is the **constant term**.

A linear equation has no terms that are products of unknowns, such as x_1^2 or x_1x_4.

Exercise C1

Which of the following are linear equations in the five unknowns $x_1, \ldots, x_5$?

(a) $x_1 + 3x_2 - x_3 - 5x_4 - 2x_5 = 0$ (b) $x_1 - x_2 + 2x_3x_4 + 3x_5 = 4$

(c) $5x_2 - x_5 = 2$ (d) $a_1x_1 + a_2x_2^2 + \cdots + a_5x_5^5 = b$

We write a system of linear equations, or more precisely *a general system of m linear equations in n unknowns*, as

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m.
\end{array}$$

The numbers b_i are the constant terms, the variables x_j are the unknowns and the numbers a_{ij} are the coefficients. We use the double subscript ij to show that a_{ij} is the coefficient of the jth unknown in the ith equation. The number m of equations need not be the same as the number n of unknowns.

A solution of a system of linear equations is a list of values for the unknowns that simultaneously satisfy each of the equations. In solving a system, we look for *all* the solutions – you have already seen that some systems have infinitely many solutions.

Definitions

The values $x_1 = c_1$, $x_2 = c_2$, ..., $x_n = c_n$ are a **solution** of a system of m linear equations in n unknowns, denoted by $x_1, \ldots, x_n$, if these values simultaneously satisfy all m equations of the system. The **solution set** of the system is the set of all the solutions.

For example, you saw earlier that the system

$$\begin{aligned} x + y + z &= 1 \\ x + y \phantom{{}+z} &= 1 \\ x \phantom{{}+y} - z &= 0, \end{aligned} \tag{1}$$

has the unique solution $x = 0$, $y = 1$, $z = 0$ corresponding to the unique point of intersection $(0, 1, 0)$ of the three planes represented by these equations. We can write the solution set of this system as the set $\{(0, 1, 0)\}$, which has just one member.

You also saw that the system

$$\begin{aligned} x + y + z &= 1 \\ x + y + z &= 2 \\ x + y - z &= 0, \end{aligned} \tag{2}$$

has no solutions, so its solution set is the empty set.

Definitions

A system of linear equations is **consistent** when it has at least one solution, and **inconsistent** when it has no solutions.

The system (1) is consistent, and the system (2) is inconsistent.

When a system of linear equations has infinitely many solutions, we can write down a general solution from which all solutions can be found as follows.

You saw earlier that the solutions of the system

$$\begin{aligned} x + y + z &= 1 \\ x + y \phantom{{}+z} &= 1 \\ x + y - z &= 1 \end{aligned}$$

are the sets of values for x, y and z satisfying $x + y = 1$ and $z = 0$. The unknowns x and y are related by the equation $x + y = 1$, which we can rewrite as $y = 1 - x$. Thus for each real parameter k assigned to the unknown x, we have a corresponding value $1 - k$ for the unknown y. We write this general solution as

$$x = k, \quad y = 1 - k, \quad z = 0, \quad \text{where } k \in \mathbb{R}.$$

To highlight the connection between the solutions of the system and the intersection of the planes in $\mathbb{R}^3$, we can write the solution set as a set of points in $\mathbb{R}^3$:

$$\{(k, 1-k, 0) \in \mathbb{R}^3 : k \in \mathbb{R}\},$$

which we usually abbreviate to

$$\{(k, 1-k, 0) : k \in \mathbb{R}\}.$$

Note that the order of the unknowns x, y, then z matters: the triples $(1, 0, 0)$ and $(0, 1, 0)$ correspond to different solutions. We could have assigned parameters differently and obtained alternative ways of writing down the solution set. For example, if we assign the real parameter p to the unknown y and rewrite the equation $x + y = 1$ as $x = 1 - y$, we get

$$\{(1-p, p, 0) : p \in \mathbb{R}\}.$$

Homogeneous systems

In the following systems of linear equations, the constant terms are all zero:

$$\begin{aligned} 2x + 3y &= 0 \\ x - y &= 0, \end{aligned} \tag{3}$$

$$\begin{aligned} x - y - z &= 0 \\ 2x + y - z &= 0 \\ -x + y + z &= 0. \end{aligned} \tag{4}$$

Such systems are called *homogeneous*.

Definitions

A **homogeneous** system of linear equations is a system of linear equations in which each constant term is zero.

A system containing at least one non-zero constant term is a **non-homogeneous** system.

If we substitute $x = 0$, $y = 0$ into system (3), and $x = 0$, $y = 0$, $z = 0$ into system (4), then all the equations are satisfied. These solutions are called *trivial*.

Definitions

The **trivial** solution to a system of homogeneous linear equations is the solution in which each unknown is equal to zero.

A solution with at least one non-zero unknown is a **non-trivial** solution.

A homogeneous system always has at least the trivial solution, and is therefore always consistent, whereas non-homogeneous systems have only non-trivial solutions or may be inconsistent.

Exercise C2

Write down a general homogeneous system of m linear equations in n unknowns, and show that the solution set contains the trivial solution.

Returning to system (4), we see that there are other solutions, unlike system (3) which has no non-trivial solutions. For example, $x = 2$, $y = -1$, $z = 3$ is a solution to system (4). In fact, this system has an infinite solution set because the first and third equations are multiples of one another. Geometrically, the three planes represented by these equations intersect in a line. Figure 7 illustrates this situation, as the planes represented by the first and third equations coincide. The solution set can be written as $\{(2k, -k, 3k) : k \in \mathbb{R}\}$.

Number of solutions

In Subsection 1.1 you saw that when $m \leq n \leq 3$, a system of m equations in n unknowns has a solution set which either

- contains exactly one solution,
- is empty, or
- contains infinitely many solutions.

(When $m = n = 1$ we have one equation of the form $ax = b$, which has a unique solution.)

In fact, as you will see in Unit C3 *Linear transformations*, the solution set of a system of m linear equations in n unknowns has one of these forms, for any natural numbers m and n.

We observed earlier that two non-parallel planes in $\mathbb{R}^3$ intersect either in a line or in a plane, so cannot intersect at a unique point. A consistent system of two linear equations in three unknowns therefore has an infinite solution set. In general, a consistent system of m equations in n unknowns, with $m < n$, has insufficient constraints on the unknowns to determine them uniquely; that is, it has an infinite solution set.

1.3 Solving systems

We now introduce a systematic method for solving systems of linear equations. This method is called **Gauss–Jordan elimination**. It entails successively transforming a system into simpler systems, in such a way that the solution set remains unchanged. The process ends when the solutions can be determined easily. You will meet this method again in Section 2, where you will use *matrices* to represent systems of linear equations. A strategy for solving systems of linear equations using Gauss–Jordan elimination is given there.

The Gauss–Jordan elimination method was introduced by the geodesist Wilhelm Jordan (1842–1899) in the third edition of his *Handbuch der Vermessungskunde* (*Handbook of Surveying*) in 1888. In the same year the rather more obscure Luxembourg mathematician turned abbot Bernard Isidore Clasen (1829–1902) independently described the method, but his work did not become widely known. The method's association with Carl Friedrich Gauss (1777–1855) is due to the fact that it can be regarded as a modification of the method of Gaussian elimination. Wilhelm Jordan is not to be confused with the algebraist Camille Jordan (1838–1922).

Wilhelm Jordan

The idea of Gauss–Jordan elimination is to reduce the number of unknowns in each equation. In general, we use the first equation to eliminate the first unknown from all the other equations, then use the second equation to eliminate another unknown (usually the second) from all the other equations, and so on. The actual order in which the unknowns are eliminated is flexible; however, it is sensible, at least initially, to proceed in order to avoid making mistakes.

To avoid confusion when applying this method, we label the current equations $\mathbf{r}_1$, $\mathbf{r}_2$, and so on. This notation will be used in Section 2 where we transform *rows* of matrices, hence the choice of the letter $\mathbf{r}$.

We can then write down how we are transforming the preceding system to obtain the current (simpler) system. We use the symbol $\leftrightarrow$ ('interchanges with') to indicate that two equations are to be interchanged; for example, $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$ means that the first and second equations are interchanged. We use the symbol $\rightarrow$ ('goes to') to show how an equation is to be transformed. For example, $\mathbf{r}_2 \rightarrow \mathbf{r}_2 + \mathbf{r}_1$ means that the second equation of the system is transformed by adding the first equation to it.

We start by illustrating this method with a system of two linear equations in two unknowns. Although this method is not the simplest way of solving this particular system, it proves very useful in solving more complicated systems. It is important that the operations we perform do not alter the solution set of the system.

Worked Exercise C1

Solve the following system of two linear equations in two unknowns.

$$\begin{aligned} 2x + 4y &= 10 \\ 4x + y &= 6 \end{aligned}$$

Solution

We aim to simplify the system by eliminating the unknown y, or y-term, from the first equation and the unknown x, or x-term, from the second; that is, we aim to obtain equations of the form $x = *$, $y = *$, where the asterisks denote numbers to be determined.

We label the two equations of the system.

$$\begin{array}{lrl} \mathbf{r}_1 & 2x + 4y &= 10 \\ \mathbf{r}_2 & 4x + y &= 6 \end{array}$$

We simplify the first equation.

We divide it through by 2, so that the coefficient of x is equal to 1.

$$\begin{array}{lrl} \mathbf{r}_1 \to \frac{1}{2}\mathbf{r}_1 & x + 2y &= 5 \\ & 4x + y &= 6 \end{array}$$

At each step, we relabel (implicitly) the equations of the current system. These two equations therefore become the *new* $\mathbf{r}_1$ and $\mathbf{r}_2$.

We then eliminate the x-term in the second equation.

We subtract 4 times the first equation from the second.

$$\begin{array}{lrl} & x + 2y &= 5 \\ \mathbf{r}_2 \to \mathbf{r}_2 - 4\mathbf{r}_1 & -\,7y &= -14 \end{array}$$

We now simplify the second equation of this new system.

We divide it through by -7; this yields a system which already looks less complicated than the original system, but has the same solution set.

$$\begin{array}{lrl} & x + 2y &= 5 \\ \mathbf{r}_2 \to -\frac{1}{7}\mathbf{r}_2 & y &= 2 \end{array}$$

Next we eliminate the y-term from the first equation.

We subtract twice the second equation from the first.

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_2 & x = 1 \\ & y = 2 \end{array}$$

We conclude that there is a unique solution: $x = 1$, $y = 2$.

As a check: we substitute $x = 1$ and $y = 2$ in the original system, using the abbreviation LHS for the left-hand side of the original equations and RHS for the right-hand side:

$$\begin{aligned} \text{LHS} &= (2 \times 1) + (4 \times 2) = 10 = \text{RHS}, \quad \checkmark \\ \text{LHS} &= (4 \times 1) + (1 \times 2) = 6 = \text{RHS}. \quad \checkmark \end{aligned}$$

The steps we performed in the worked exercise above involve either multiplying (or dividing) an equation by a non-zero number, or changing one equation by adding (or subtracting) a multiple of another. Neither of these operations alters the solution set of the system. Changing the order in which we write down the equations also does not alter the solution set of the system. These are the three operations, called *elementary operations*, that we perform to simplify a system of linear equations when using the method of Gauss–Jordan elimination.

Elementary operations

The following operations do not change the solution set of a system of linear equations.

1. Interchange two equations.
2. Multiply an equation by a non-zero number.
3. Change one equation by adding to it a multiple of another.

Operation 2 includes division by a non-zero number, and operation 3 includes subtracting a multiple of one equation from another.

In symbols we represent these three elementary operations by

$$\mathbf{r}_i \leftrightarrow \mathbf{r}_j, \quad \mathbf{r}_i \to \alpha\mathbf{r}_i, \quad \text{and} \quad \mathbf{r}_i \to \mathbf{r}_i + \beta\mathbf{r}_j,$$

respectively, where α, β are non-zero numbers.

Exercise C3

Perform elementary operations, as in Worked Exercise C1, to solve the following system of two linear equations in two unknowns.

$$\begin{aligned} x + y &= 4 \\ 2x - y &= 5 \end{aligned}$$

We now solve a system of three linear equations in three unknowns. We use elementary operations to try to reduce the system to the following form, where again the asterisks denote numbers to be determined.

$$\begin{aligned} x &= * \\ y &= * \\ z &= * \end{aligned}$$

Worked Exercise C2

Solve the following system of three linear equations in three unknowns.

$$\begin{aligned} x + y + 2z &= 3 \\ 2x + 2y + 3z &= 5 \\ x - y &= 5 \end{aligned}$$

Solution

We label the three equations and apply elementary operations to simplify the system.

$$\begin{array}{ll} \mathbf{r}_1 & x + y + 2z = 3 \\ \mathbf{r}_2 & 2x + 2y + 3z = 5 \\ \mathbf{r}_3 & x - y = 5 \end{array}$$

We eliminate the x-term from the second and third equations: we subtract twice the first equation from the second, and then subtract the first from the third.

$$\begin{array}{ll} & x + y + 2z = 3 \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 & -z = -1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1 & -2y - 2z = 2 \end{array}$$

We now have no y-term in the second equation, and so cannot use this equation to eliminate the y-term from the first and third equations. We also cannot use the first equation to eliminate the y-term from the third equation, as this would reintroduce an x-term. However, we can use the third equation to eliminate the y-term from the first equation. To keep the terms in order we interchange the second and third equations – this is not strictly necessary, but helps keep things in order.

$$\begin{array}{ll} & x + y + 2z = 3 \\ \mathbf{r}_2 \leftrightarrow \mathbf{r}_3 & -2y - 2z = 2 \\ & -z = -1 \end{array}$$

We simplify this new second equation by dividing it through by -2.

$$\begin{array}{ll} & x + y + 2z = 3 \\ \mathbf{r}_2 \to -\frac{1}{2}\mathbf{r}_2 & y + z = -1 \\ & -z = -1 \end{array}$$

We eliminate the y-term from the first equation by subtracting the second equation from the first.

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 & x + z = 4 \\ & y + z = -1 \\ & -z = -1 \end{array}$$

We simplify the third equation by multiplying it through by -1.

$$\begin{array}{ll} & x \quad + z = 4 \\ & \quad y + z = -1 \\ \mathbf{r}_3 \to -\mathbf{r}_3 & \qquad z = 1 \end{array}$$

Finally, we eliminate the z-term from the first and second equations by subtracting the third equation from both the first and second equations.

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_3 & x = 3 \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_3 & y = -2 \\ & z = 1 \end{array}$$

We conclude that there is a unique solution: $x = 3$, $y = -2$, $z = 1$.

As a check: we substitute $x = 3$, $y = -2$ and $z = 1$ into the original system:

$$\begin{aligned} \text{LHS} &= (1 \times 3) + (1 \times (-2)) + (2 \times 1) = 3 = \text{RHS}, \checkmark \\ \text{LHS} &= (2 \times 3) + (2 \times (-2)) + (3 \times 1) = 5 = \text{RHS}, \checkmark \\ \text{LHS} &= (1 \times 3) - (1 \times (-2)) + (0 \times 1) = 5 = \text{RHS}. \checkmark \end{aligned}$$

Exercise C4

Solve the following system of three linear equations in three unknowns.

$$\begin{aligned} x + y - z &= 8 \\ 2x - y + z &= 1 \\ -x + 3y + 2z &= -8 \end{aligned}$$

Each system solved so far in this subsection has a unique solution. We now show how to apply the method to a system that does not have a unique solution.

It is not usually possible to reduce a system with an infinite solution set to one where each equation contains just one unknown. This is illustrated by the following worked exercise.

Worked Exercise C3

Solve the following system of three linear equations in three unknowns.

$$\begin{aligned} x + 2y \phantom{{}- z} &= 0 \\ y - z &= 2 \\ x + y + z &= -2 \end{aligned}$$

Solution

We label the three equations and apply elementary operations to simplify the system.

$$\begin{array}{lrcrcrcl} \mathbf{r}_1 & x & + & 2y & & & = & 0 \\ \mathbf{r}_2 & & & y & - & z & = & 2 \\ \mathbf{r}_3 & x & + & y & + & z & = & -2 \end{array}$$

We eliminate the x-term from $\mathbf{r}_3$ using $\mathbf{r}_1$.

$$\begin{array}{lrcrcrcl} & x & + & 2y & & & = & 0 \\ & & & y & - & z & = & 2 \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1 & & & -y & + & z & = & -2 \end{array}$$

We eliminate the y-terms from $\mathbf{r}_1$ and $\mathbf{r}_3$ using $\mathbf{r}_2$.

$$\begin{array}{lrcrcrcl} \mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_2 & x & & & + & 2z & = & -4 \\ & & & y & - & z & = & 2 \\ \mathbf{r}_3 \to \mathbf{r}_3 + \mathbf{r}_2 & 0x & + & 0y & + & 0z & = & 0 \end{array}$$

The current $\mathbf{r}_3$ equation gives no constraints on x, y and z: any values for x, y and z satisfy it.

If we were to try to use equation $\mathbf{r}_2$ to eliminate the z-term from $\mathbf{r}_1$, we would introduce a y-term. Similarly, using equation $\mathbf{r}_1$ to eliminate the z-term from the equation $\mathbf{r}_2$ would reintroduce an x-term.

There are insufficient constraints on the unknowns to determine them uniquely; so the system has an infinite solution set.

We have two equations, one ($x = -4 - 2z$) relating the unknowns x and z, and the other ($y = 2 + z$) relating y and z.

As each equation involves a z-term, we can choose any value we wish for z and use the equations to find the corresponding values for x and y in terms of this value for z. We set z equal to the real parameter k to get a general solution.

We write the general solution as

$$x = -4 - 2k, \quad y = 2 + k, \quad z = k, \quad k \in \mathbb{R}.$$

In the worked exercise above the equation $\mathbf{r}_3$ was written as $0x + 0y + 0z = 0$ to highlight the fact that all the coefficients are zero – in future we will simply write the equivalent equation $0 = 0$. In this case, the original equation $\mathbf{r}_3$ did not give rise to any additional constraints not already given by $\mathbf{r}_1$ and $\mathbf{r}_2$.

Whenever the simplification results in an equation $0 = 0$, we have, in effect, reduced the number of equations. We simplify the remaining equations as far as possible, in order to determine the solution set.

Exercise C5

Solve the following system of three linear equations in three unknowns.

$$\begin{aligned} x + 3y - z &= 4 \\ -x + 2y - 4z &= 6 \\ x + 2y \phantom{{}-4z} &= 2 \end{aligned}$$

We now try to solve an inconsistent system.

Worked Exercise C4

Solve the following system of three linear equations in three unknowns.

$$\begin{aligned} x + 2y + 4z &= 6 \\ y + z &= 1 \\ x + 3y + 5z &= 10 \end{aligned}$$

Solution

We label the three equations and apply elementary operations to simplify the system.

$$\begin{array}{ll} \mathbf{r}_1 & x + 2y + 4z = 6 \\ \mathbf{r}_2 & y + z = 1 \\ \mathbf{r}_3 & x + 3y + 5z = 10 \end{array}$$

We eliminate the x-term from $\mathbf{r}_3$ using $\mathbf{r}_1$.

$$\begin{array}{ll} & x + 2y + 4z = 6 \\ & y + z = 1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1 & y + z = 4 \end{array}$$

Comparing $\mathbf{r}_2$ and $\mathbf{r}_3$, we can conclude at this point that the system is inconsistent or we can carry out one further step to eliminate the y-terms from $\mathbf{r}_1$ and $\mathbf{r}_3$ using $\mathbf{r}_2$.

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_2 & x \phantom{{}+y} + 2z = 4 \\ & \phantom{x+{}} y + z = 1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_2 & 0 = 3 \end{array}$$

Concentrating on the current $\mathbf{r}_3$ equation $(0 = 3)$, we see that there are no values of x, y and z that satisfy it. This system has no solutions.

This system of linear equations is inconsistent: the solution set is the empty set.

Whenever the simplification results in an equation $0 = *$, where the asterisk $*$ denotes a non-zero number, we have an inconsistent system, since such an equation has no solutions. There is no point in simplifying the remaining equations further. As indicated in Worked Exercise C4, inconsistency of the system could have been inferred at the penultimate stage, as the equations $y + z = 1$ and $y + z = 4$ form an inconsistent system.

Exercise C6

Solve the following system of three linear equations in three unknowns.

$$\begin{aligned} x + y + z &= 6 \\ -x + y - 3z &= -2 \\ 2x + y + 3z &= 6 \end{aligned}$$

1.4 Applications

Systems of linear equations frequently arise when we use mathematics to solve problems from both within mathematics and outside it.

The following worked exercise illustrates how linear equations can be used to find the equation of a plane through three given points.

Worked Exercise C5

Determine the equation of the plane that contains the three points $(1, 3, 1)$, $(1, 5, 2)$ and $(2, 2, 1)$.

Solution

Let the equation of the plane be

$$ax + by + cz = d,$$

where a, b, c and d are real, and a, b and c are not all zero.

Each of the points is contained in the plane, and so each set of coordinates satisfies this equation.

We do not specify a value for d at this point because we can multiply or divide the equation through by a constant without affecting the plane it represents.

Substituting the points into the equation gives a system of three linear equations in the unknowns a, b and c. We label the equations and apply elementary operations to simplify the system.

$$\begin{array}{ll} \mathbf{r}_1 & a + 3b + c = d \\ \mathbf{r}_2 & a + 5b + 2c = d \\ \mathbf{r}_3 & 2a + 2b + c = d \end{array}$$

We eliminate the a-term from $\mathbf{r}_2$ and $\mathbf{r}_3$ using $\mathbf{r}_1$.

$$\begin{array}{ll} & a + 3b + c = d \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_1 & \quad 2b + c = 0 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 2\mathbf{r}_1 & \;- 4b - c = -d \end{array}$$

We simplify $\mathbf{r}_2$.

$$\begin{array}{ll} & a + 3b + \phantom{\tfrac{1}{2}}c = d \\ \mathbf{r}_2 \to \frac{1}{2}\mathbf{r}_2 & \qquad b + \tfrac{1}{2}c = 0 \\ & \;- 4b - \phantom{\tfrac{1}{2}}c = -d \end{array}$$

We eliminate the b-term from $\mathbf{r}_1$ and $\mathbf{r}_3$ using $\mathbf{r}_2$.

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - 3\mathbf{r}_2 & a \quad - \tfrac{1}{2}c = d \\ & \quad b + \tfrac{1}{2}c = 0 \\ \mathbf{r}_3 \to \mathbf{r}_3 + 4\mathbf{r}_2 & \qquad\quad c = -d \end{array}$$

We eliminate the c-term from $\mathbf{r}_1$ and $\mathbf{r}_2$ using $\mathbf{r}_3$.

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 + \frac{1}{2}\mathbf{r}_3 & a = \tfrac{1}{2}d \\ \mathbf{r}_2 \to \mathbf{r}_2 - \frac{1}{2}\mathbf{r}_3 & b = \tfrac{1}{2}d \\ & c = -d \end{array}$$

We conclude that this system has a unique solution (in terms of d): $a = \frac{1}{2}d,\ b = \frac{1}{2}d,\ c = -d$.

We substitute these expressions into the equation of the plane to get

$$\tfrac{1}{2}dx + \tfrac{1}{2}dy - dz = d.$$

Multiplying through by 2 and dividing through by d yields a simpler equation for the plane:

$$x + y - 2z = 2.$$

As a check: we substitute the coordinates of each of the three points into this equation for the plane

$$\begin{aligned} \text{LHS} &= (1 \times 1) + (1 \times 3) - (2 \times 1) = 2 = \text{RHS}, \checkmark \\ \text{LHS} &= (1 \times 1) + (1 \times 5) - (2 \times 2) = 2 = \text{RHS}, \checkmark \\ \text{LHS} &= (1 \times 2) + (1 \times 2) - (2 \times 1) = 2 = \text{RHS}. \checkmark \end{aligned}$$

Exercise C7

Determine the equation of the plane that contains the three points $(1, 0, 2)$, $(0, 3, 4)$ and $(1, 1, 3)$.

The final exercise in this section uses systems of linear equations to solve a different type of problem. The idea is to use the information given to write down two linear equations that simultaneously hold, and then to solve these to answer the question.

Exercise C8

The sum of the ages of my sister and my brother is 40 years. My brother is 12 years older than my sister. How old is my sister?

Because Gauss–Jordan elimination is a systematic method for solving systems of linear equations, it is straightforward to automate. Hence large systems of linear equations involving many variables can be easily solved using computers. Such systems are used in some methods of weather forecasting, as well as systems of non-linear equations. Gauss–Jordan elimination also arises in coding theory, which underpins digital communication and data transmission.

2 Row-reduction

In this section you will see how the method of Gauss–Jordan elimination can be applied using matrices, and that it can be formalised into a strategy that can be followed step by step. This method makes it easy to solve even quite large systems of linear equations. It involves a technique (*row-reduction*) that will be useful in another context later in this unit when we look at inverses of matrices.

2.1 Augmented matrices

We begin by using *matrices* as an abbreviated notation for a system of linear equations.

A **matrix** is simply a rectangular array of objects, usually numbers, enclosed in brackets; in this module we use round brackets for matrices, although some texts use square ones.

The objects in a matrix are called its **entries**. The entries along a horizontal line form a **row**, and those down a vertical line form a **column**. A matrix with m rows and n columns is an $m \times n$ matrix, and we say that it is a matrix of **size** $m \times n$.

A **zero row** of a matrix is a row comprising entirely of zeros, and a **non-zero row** has at least one non-zero entry. The first non-zero entry in a row (reading from left to right) of a matrix is the **leading entry** of that row; when such an entry in a row is the number 1, it is called a **leading 1**.

Here are some examples of matrices with some entries highlighted as explained below:

$$\begin{pmatrix} 2 & -7 \\ -1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3.17 & 2.23 & 7.05 & 0.00 \\ 4.88 & 1.71 & 1.72 & 5.55 \end{pmatrix}, \quad \begin{pmatrix} 3 & 7 & 12 \\ 0 & 1 & 8 \\ 0 & 0 & -5 \end{pmatrix}.$$

The entries in the first row of the first matrix above are 2 and -7; the entries in the second column of the second matrix above are 2.23 and 1.71; the 1 in the second row of the third matrix is a leading 1, and the -5 in the third row of this matrix is a leading entry.

We can abbreviate a system of linear equations by writing its coefficients and constants in the form of a matrix. For example, the system

$$\begin{aligned} 4x + y &= -7 \\ x - 3y &= 0 \end{aligned}$$

can be abbreviated as

$$\left(\begin{array}{cc|c} 4 & 1 & -7 \\ 1 & -3 & 0 \end{array}\right).$$

It is helpful to draw the vertical line separating the coefficients of the unknowns on the left-hand sides of the equations from the constants on the right-hand sides.

In general, the system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

of m linear equations in n unknowns $x_1, x_2, \dots, x_n$ is abbreviated as the matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right).$$

This matrix is called the **augmented matrix** of the system. The word *augmented* reflects the fact that it is made up of a matrix formed by the coefficients of the unknowns on the left-hand sides of the equations, *augmented* by a matrix (or column vector) formed by the constants on the right-hand sides. Later, we will sometimes consider these two matrices separately.

In the augmented matrix each row corresponds to an equation, and each column (except the last) corresponds to an unknown, in the sense that it contains all the coefficients of that unknown from the various equations. The last column corresponds to the right-hand sides of the equations.

Worked Exercise C6

Write down the augmented matrix of the following system of linear equations.

$$\begin{aligned} x \phantom{{}+ y} + 10z &= 5 \\ 3x + y - 4z &= -1 \\ 4x - 2y + 6z &= 1 \end{aligned}$$

Solution

Before writing down the augmented matrix of a system of linear equations, we must ensure that the unknowns appear in the same order in each equation, with gaps left for 'missing' unknowns (that is, unknowns whose coefficient is 0).

The augmented matrix of the system is

$$\left(\begin{array}{rrr|r} 1 & 0 & 10 & 5 \\ 3 & 1 & -4 & -1 \\ 4 & -2 & 6 & 1 \end{array}\right).$$

Worked Exercise C7

Write down the system of linear equations corresponding to the following augmented matrix, given that the unknowns are, in order, x_1, x_2.

$$\left(\begin{array}{rr|r} 1 & -2 & 5 \\ 0 & 1 & 9 \\ 4 & 3 & 0 \end{array}\right)$$

Solution

The corresponding system is

$$\begin{aligned} x_1 - 2x_2 &= 5 \\ x_2 &= 9 \\ 4x_1 + 3x_2 &= 0. \end{aligned}$$

Exercise C9

(a) Write down the augmented matrix of the following system of linear equations.

$$\begin{aligned} 4x_1 - 2x_2 \qquad\quad &= -7 \\ x_2 + 3x_3 &= 0 \\ -3x_2 + \ x_3 &= 3 \end{aligned}$$

(b) Write down the system of linear equations corresponding to the following augmented matrix, given that the unknowns are, in order, x, y, z, w.

$$\left(\begin{array}{rrrr|r} 2 & 3 & 0 & 7 & 1 \\ 0 & 1 & -7 & 0 & -1 \\ 1 & 0 & 3 & -1 & 2 \end{array}\right)$$

2.2 Elementary row operations

When you used Gauss–Jordan elimination to solve a system of linear equations in Section 1, you worked directly with the system itself; but it is often easier to apply the same method to its abbreviated form, the augmented matrix. The three elementary operations on the equations of the system correspond exactly to three equivalent operations on the rows of its augmented matrix.

Recall that the three elementary operations are as follows.

1. Interchange two equations.
2. Multiply an equation by a non-zero number.
3. Change one equation by adding to it a multiple of another.

These correspond to the following operations on the rows of the augmented matrix.

Elementary row operations

1. Interchange two rows.
2. Multiply a row by a non-zero number.
3. Change one row by adding to it a multiple of another.

We call these operations the **elementary row operations** of types 1, 2 and 3, respectively.

The next worked exercise shows a system of linear equations solved by Gauss–Jordan elimination. In Worked Exercise C2 we solved this system by performing elementary operations on the system itself; here we perform the corresponding elementary row operations on the augmented matrix of the system. You can see that here we have less to write down at each stage.

In this worked exercise, and elsewhere, we use the same notation for elementary row operations as we use for elementary operations ($\mathbf{r}_i \leftrightarrow \mathbf{r}_j$, and so on).

Worked Exercise C8

Solve the following system of linear equations.

$$\begin{aligned} x + y + 2z &= 3 \\ 2x + 2y + 3z &= 5 \\ x - y \phantom{{}+3z} &= 5 \end{aligned}$$

Solution

We perform a sequence of elementary row operations on the augmented matrix of the system.

The idea is to transform the augmented matrix into one of a system with the same solution set but whose solution set is easy to write down.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \qquad \left(\begin{array}{rrr|r} 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & 5 \\ 1 & -1 & 0 & 5 \end{array}\right)$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1 \end{array} \qquad \left(\begin{array}{rrr|r} 1 & 1 & 2 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & -2 & -2 & 2 \end{array}\right)$$

$$\begin{array}{l} \\ \mathbf{r}_2 \leftrightarrow \mathbf{r}_3 \\ \\ \end{array} \qquad \left(\begin{array}{rrr|r} 1 & 1 & 2 & 3 \\ 0 & -2 & -2 & 2 \\ 0 & 0 & -1 & -1 \end{array}\right)$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to -\tfrac{1}{2}\mathbf{r}_2 \\ \\ \end{array} \qquad \left(\begin{array}{rrr|r} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & -1 \end{array}\right)$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 \\ \\ \\ \end{array} \qquad \left(\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & -1 \end{array}\right)$$

$$\begin{array}{l} \\ \\ \mathbf{r}_3 \to -\mathbf{r}_3 \end{array} \qquad \left(\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_3 \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_3 \\ \\ \end{array} \qquad \left(\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

The corresponding system is

$$\begin{aligned} x \phantom{{}+y+z} &= 3 \\ y \phantom{{}+z} &= -2 \\ z &= 1. \end{aligned}$$

The unique solution is $x = 3$, $y = -2$, $z = 1$.

It is important to appreciate the following point about elementary row operations.

When a sequence of elementary row operations is performed on a matrix, each row operation in the sequence produces a new matrix, and the following row operation is then performed on that new matrix. For example, the working for the first two row operations in the solution to the

worked exercise above should, strictly, be set out as follows.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & 5 \\ 1 & -1 & 0 & 5 \end{array}\right)$$

$$\mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 \qquad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 5 \end{array}\right)$$

$$\mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1 \qquad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & -2 & -2 & 2 \end{array}\right)$$

However, we often perform two or more row operations in one step, to save time. Whenever we do this, we must ensure that when a row is changed by one of these row operations, the *new version* of that row is used when performing later row operations.

The easiest way to avoid difficulties is to perform two or more row operations in one step *only if none of these row operations changes a row that is then used by another of these row operations*. The above row operations $\mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1$ and $\mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1$ meet this criterion: the first changes only row 2, and the second does not involve row 2. In this module we perform two or more row operations in one step only if they meet this criterion.

Row-sum check

We end this subsection by describing a simple checking method that can be useful for picking up arithmetical errors when we perform a sequence of elementary row operations on a matrix by hand.

To apply this method, we proceed as follows. To the right of each row of the initial matrix, we write down the sum of the entries in that row.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & 5 \\ 1 & -1 & 0 & 5 \end{array}\right) \quad \begin{array}{rl} 7 & (= 1+1+2+3) \\ 12 & (= 2+2+3+5) \\ 5 & (= 1-1+5) \end{array}$$

From then on, when performing elementary row operations, we treat this 'check column' of numbers as if it were an extra column of the matrix, and perform the row operations on it. So the first step of the calculation in the solution to Worked Exercise C8 above would look as follows.

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1 \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & -2 & -2 & 2 \end{array}\right) \quad \begin{array}{rl} 7 & \\ -2 & (= 12 - 2 \times 7) \\ -2 & (= 5 - 7) \end{array}$$

At each step in the calculation, each entry in this extra column should still be the sum of the entries in the corresponding row. If this is not the case, then an error has been made.

2.3 Solving linear equations systematically

We now describe a systematic method for solving a system of linear equations by Gauss–Jordan elimination. The method involves performing elementary row operations on the augmented matrix of the system. In fact you have already seen this method in action, in Worked Exercise C8. Here we detail the sequence of steps involved, setting out a general strategy.

The strategy involves writing down the augmented matrix of the system of equations and then performing a sequence of elementary row operations that reduces it to a simpler form called *row-reduced form*. The system of equations corresponding to this row-reduced matrix has the same solution set as the original system but with the new system it is easy to work out what the solution set is. The process of reducing the matrix to row-reduced form is referred to as *row-reduction*. We start by describing what a row-reduced matrix looks like.

Row-reduced matrices

Here is an example of a row-reduced matrix.

$$\begin{pmatrix} 0 & 1 & 6 & 0 & -5 & 3 & 0 & 0 & 17 \\ 0 & 0 & 0 & 1 & 3 & 24 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The entries of a row-reduced matrix have a staircase form which we have emphasised with a black line. All the entries below the staircase must be 0; the left-most entry on each line above the staircase must be a 1 and all the other entries in that column must be 0. The other entries above the staircase can be any numbers.

The general form of a row-reduced matrix is illustrated below, where the entries not in a column containing a leading 1 are indicated with asterisks, and the fact that all the entries below the staircase are zero is indicated by the large zero.

$$\begin{pmatrix} & 1 & *\cdots* & 0 & *\cdots* & 0 & *\cdots* & 0 & *\cdots* \\ & & & 1 & *\cdots* & 0 & *\cdots* & 0 & *\cdots* \\ & & & & & 1 & *\cdots* & 0 & *\cdots* \\ & & \text{\Large 0} & & & & & 1 & *\cdots* \\ & & & & & & & & \end{pmatrix}$$

We can describe a row-reduced matrix more precisely by specifying that it must satisfy certain properties as in the definition below.

Definition

A **row-reduced matrix** is a matrix satisfying the following four properties.

1. Any zero rows are at the bottom of the matrix.
2. Each non-zero row has a leading 1.
3. Each leading 1 is to the right of the leading 1 in the row above.
4. Each leading 1 is the only non-zero entry in its column.

Property 3 gives a row-reduced matrix its staircase form, and property 4 ensures that the entries above and below a leading 1 are all 0.

Here are some more examples of row-reduced matrices:

$$\begin{pmatrix} 1 & 0 & \frac{1}{4} & 7 & 0 & 23 & 0 \\ 0 & 1 & \frac{3}{4} & -8 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

However, none of the following matrices are row-reduced:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 3 & 5 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The first matrix is not row-reduced as it has neither property 1 nor property 2. The second is not row-reduced as it does not have property 3; the third does not have property 4.

Exercise C10

Which of the following are row-reduced matrices?

(a) $\begin{pmatrix} 0 & 1 & 7 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 7 \\ 0 & 0 & 1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Finding solutions from row-reduced form

Suppose we have been given a system of linear equations, and that we have written down its augmented matrix and performed a sequence of elementary row operations to reduce it to row-reduced form. We will describe these operations in detail shortly, but first we will consider how to find all the solutions from the row-reduced form.

Unique solution

Consider for example this row-reduced augmented matrix:

$$\left(\begin{array}{ccc|r} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

Suppose the unknowns are x_1, x_2 and x_3. Then the system of equations corresponding to this matrix is as follows.

$$\begin{aligned} x_1 \qquad\quad &= 8 \\ x_2 \qquad &= 3 \\ x_3 &= -1 \end{aligned}$$

This system is already in solved form, and we can immediately write down the unique solution:

$$x_1 = 8,\ x_2 = 3,\ x_3 = -1.$$

Unique solution

Whenever the original system of equations has a unique solution it can be written down directly from the row-reduced matrix.

No solution

Now consider this row-reduced augmented matrix:

$$\left(\begin{array}{crc|c} 1 & -6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

We write down the system of equations corresponding to the matrix.

$$\begin{aligned} x_1 - 6x_2 \qquad &= 0 \\ x_3 &= 0 \\ 0 &= 1 \end{aligned}$$

This time we find that one of the equations is $0 = 1$. This equation cannot hold, so it follows that the system of equations has no solutions; that is, the equations are inconsistent.

No solution

Whenever the original system of equations is inconsistent, row-reducing the augmented matrix yields a system that includes the equation $0 = 1$.

Infinitely many solutions

Now consider this row-reduced augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 6 & 7 \\ 0 & 1 & -4 & 2 \end{array}\right).$$

We can write down the system of equations corresponding to this row-reduced matrix, but this time the system does not immediately give us a solution nor does it include the equation $0 = 1$.

$$\begin{aligned} x_1 \qquad + 6x_3 &= 7 \\ x_2 - 4x_3 &= 2 \end{aligned}$$

In this case we rearrange each equation so that everything except the first term on the left is moved over to the right-hand side. The effect of this is to express each **leading unknown**, that is, each unknown that corresponds to a column containing a leading 1, in terms of the other unknowns, the **non-leading unknowns**. Here, x_1 and x_2 are leading unknowns and x_3 is the only non-leading unknown.

$$\begin{aligned} x_1 &= 7 - 6x_3 \\ x_2 &= 2 + 4x_3 \end{aligned}$$

Having expressed the two leading unknowns, x_1 and x_2, in terms of the non-leading unknown x_3 we can then choose any value we like for x_3 and the equations give us the corresponding values of x_1 and x_2. So the system has infinitely many solutions – one for each choice of value of x_3. If we set $x_3 = k$ ($k \in \mathbb{R}$), say, and substitute this into the expressions for x_1 and x_2, then we have all the unknowns expressed in terms of the parameter k. The general solution of the system is therefore:

$$\begin{aligned} x_1 &= 7 - 6k \\ x_2 &= 2 + 4k \\ x_3 &= k \quad (k \in \mathbb{R}). \end{aligned}$$

Infinitely many solutions

Whenever the original system of equations has infinitely many solutions, the general solution can be determined by setting the non-leading unknowns equal to parameters and expressing all the unknowns in terms of these parameters.

As we noted in Subsection 1.2, a system of linear equations must have one of these three possibilities: a unique solution, no solution, or infinitely many solutions.

Worked Exercise C9

Solve the system corresponding to the following row-reduced augmented matrix. Assume that the unknowns are x_1, x_2, x_3, x_4, x_5.

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 5 & 4 \\ 0 & 1 & -3 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 3 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Solution

We write down the system of equations, ignoring the equation corresponding to the bottom row of the matrix since it is just $0 = 0$.

The augmented matrix corresponds to the system

$$\begin{aligned} x_1 \qquad + 2x_3 \qquad + 5x_5 &= 4 \\ x_2 - 3x_3 \qquad - \; x_5 &= 2 \\ x_4 + 3x_5 &= -7. \end{aligned}$$

The system does not immediately give us a solution, and so there is not a unique solution. Furthermore, there is no equation $0 = 1$ and so the system is not inconsistent. Therefore it must be a system with infinitely many solutions. We express each leading unknown, x_1, x_2 and x_4, in terms of the non-leading unknowns, x_3 and x_5.

This system is equivalent to

$$\begin{aligned} x_1 &= 4 - 2x_3 - 5x_5 \\ x_2 &= 2 + 3x_3 + x_5 \\ x_4 &= -7 - 3x_5. \end{aligned}$$

We can choose any values for x_3 and x_5, and the equations will give us the corresponding values of x_1, x_2 and x_4. So the system does have infinitely many solutions, one for each choice of values for x_3 and x_5. To obtain the general solution of the system, we set x_3 and x_5 equal to parameters.

Setting $x_3 = k$ and $x_5 = l$, $(k, l \in \mathbb{R})$, we obtain the general solution

$$\begin{aligned} x_1 &= 4 - 2k - 5l \\ x_2 &= 2 + 3k + l \\ x_3 &= k \\ x_4 &= -7 - 3l \\ x_5 &= l \quad (k, l \in \mathbb{R}). \end{aligned}$$

Exercise C11

Solve the system corresponding to each of the following row-reduced augmented matrices.

(a) Assume that the unknowns are x_1, x_2.

$$\left(\begin{array}{cc|c} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} \end{array}\right)$$

(b) Assume that the unknowns are x_1, x_2, x_3.

$$\left(\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

(c) Assume that the unknowns are x_1, x_2, x_3, x_4, x_5.

$$\left(\begin{array}{ccccc|r} 1 & 3 & 0 & 2 & 0 & -7 \\ 0 & 0 & 1 & -3 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(d) Assume that the unknowns are x_1, x_2, x_3, x_4.

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 4 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

Row-reduction strategy

You have seen that, once the augmented matrix of a system of linear equations is reduced to row-reduced form, all the solutions of the system can easily be determined. You will now see that there is a systematic strategy for row-reducing *any* matrix using elementary row operations.

The idea of the strategy is to take each row of the matrix in turn as the current row, starting with the first. With row 1 as the current row, we carry out four steps, then with row 2 as the current row we carry out the same four steps, and so on. In outline the steps are as follows. In step 1 we identify the column for the current row's leading 1; in steps 2 and 3 we create a leading 1 in the current row. Finally in step 4 we make each entry above and below the leading 1 into a 0.

Strategy C1

To row-reduce a matrix using elementary row operations, carry out the following four steps, first with row 1 as the current row, then with row 2 as the current row, and so on, until

- **either** every row has been the current row,
- **or** step 1 is not possible.

1. Select the first column from the left that has at least one non-zero entry in or below the current row.
2. If the current row has a 0 in the selected column, interchange it with a row below that has a non-zero entry in that column.
3. If the entry in the current row and the selected column is c, multiply the current row by $1/c$ to create a leading 1.
4. Add suitable multiples of the current row to the other rows to make each entry above and below the leading 1 into a 0.

The strategy is illustrated in the following worked exercise, where the selected rows and columns of the matrix are highlighted by shading.

You do not need to include this level of detail in your solutions, or the shading, just the row operations and current matrix. It is always a good idea to include the check column to try to pick up any errors in arithmetic!

Worked Exercise C10

Use Strategy C1 to row-reduce the following matrix.

$$\begin{pmatrix} 2 & 4 & -2 & 2 & 4 \\ 3 & 6 & -3 & 6 & 5 \\ 2 & 1 & -11 & 2 & 6 \\ -1 & 1 & 10 & -7 & -2 \end{pmatrix}$$

Solution

Row 1 is the current row.

Step 1 identifies the column for the current row's leading 1: column 1. It is the first column from the left with at least one non-zero entry in, or below, the current row.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \end{array} \quad \begin{pmatrix} 2 & 4 & -2 & 2 & 4 \\ 3 & 6 & -3 & 6 & 5 \\ 2 & 1 & -11 & 2 & 6 \\ -1 & 1 & 10 & -7 & -2 \end{pmatrix} \begin{array}{r} 10 \\ 17 \\ 0 \\ 1 \end{array}$$

Steps 2 and 3 create a leading 1 in the current row.

The current row does not have a 0 in the column selected: it has a 2, and so step 2 does not apply. In step 3 we multiply the current row by the reciprocal of 2; that is, by $\frac{1}{2}$.

In fact, when using this strategy to row-reduce a matrix there is often nothing to be done in step 2.

$$\begin{array}{l} \mathbf{r}_1 \to \frac{1}{2}\mathbf{r}_1 \\ \\ \\ \\ \end{array} \qquad \begin{pmatrix} 1 & 2 & -1 & 1 & 2 \\ 3 & 6 & -3 & 6 & 5 \\ 2 & 1 & -11 & 2 & 6 \\ -1 & 1 & 10 & -7 & -2 \end{pmatrix} \begin{array}{r} 5 \\ 17 \\ 0 \\ 1 \end{array}$$

Step 4 makes each entry above and below the current leading 1 into a 0 by adding suitable multiples of the current row to the other rows.

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 3\mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 2\mathbf{r}_1 \\ \mathbf{r}_4 \to \mathbf{r}_4 + \mathbf{r}_1 \end{array} \qquad \begin{pmatrix} 1 & 2 & -1 & 1 & 2 \\ 0 & 0 & 0 & 3 & -1 \\ 0 & -3 & -9 & 0 & 2 \\ 0 & 3 & 9 & -6 & 0 \end{pmatrix} \begin{array}{r} 5 \\ 2 \\ -10 \\ 6 \end{array}$$

None of these row operations changes a row that is then used by another of these row operations, so they can be carried out in one go; in fact, this will always be the case for the row operations required in step 4.

Row 2 is the current row.

Step 1 identifies the column for the current row's leading 1: column 2.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \end{array} \qquad \begin{pmatrix} 1 & 2 & -1 & 1 & 2 \\ 0 & 0 & 0 & 3 & -1 \\ 0 & -3 & -9 & 0 & 2 \\ 0 & 3 & 9 & -6 & 0 \end{pmatrix} \begin{array}{r} 5 \\ 2 \\ -10 \\ 6 \end{array}$$

Steps 2 and 3 create a leading 1 in the current row.

The current row does have a 0 in the column selected, so in step 2 we interchange it with a row below that has a non-zero entry in that column. We choose to interchange it with row 4, although it does not matter which row of these we use.

$$\begin{array}{l} \\ \mathbf{r}_2 \leftrightarrow \mathbf{r}_4 \\ \\ \\ \end{array} \qquad \begin{pmatrix} 1 & 2 & -1 & 1 & 2 \\ 0 & 3 & 9 & -6 & 0 \\ 0 & -3 & -9 & 0 & 2 \\ 0 & 0 & 0 & 3 & -1 \end{pmatrix} \begin{array}{r} 5 \\ 6 \\ -10 \\ 2 \end{array}$$

The current row has a 3 in the column selected, so in step 3 we multiply the current row by the reciprocal of 3; that is, $\frac{1}{3}$.

$$\begin{array}{l} \\ \mathbf{r}_2 \to \frac{1}{3}\mathbf{r}_2 \\ \\ \\ \end{array} \qquad \left(\begin{array}{rrrrr} 1 & 2 & -1 & 1 & 2 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & -3 & -9 & 0 & 2 \\ 0 & 0 & 0 & 3 & -1 \end{array}\right) \quad \begin{array}{r} 5 \\ 2 \\ -10 \\ 2 \end{array}$$

Step 4 makes each entry above and below the current leading 1 into a 0 by adding suitable multiples of the current row to the other rows.

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 + 3\mathbf{r}_2 \\ \\ \end{array} \qquad \left(\begin{array}{rrrrr} 1 & 0 & -7 & 5 & 2 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 3 & -1 \end{array}\right) \quad \begin{array}{r} 1 \\ 2 \\ -4 \\ 2 \end{array}$$

Row 3 is the current row.

Step 1 identifies the column for the current row's leading 1: column 4.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \end{array} \qquad \left(\begin{array}{rrrrr} 1 & 0 & -7 & 5 & 2 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 3 & -1 \end{array}\right) \quad \begin{array}{r} 1 \\ 2 \\ -4 \\ 2 \end{array}$$

Steps 2 and 3 create a leading 1 in the current row.

Here, step 2 does not apply, and in step 3 we multiply the current row by the reciprocal of -6; that is, $-\frac{1}{6}$.

$$\begin{array}{l} \\ \\ \mathbf{r}_3 \to -\frac{1}{6}\mathbf{r}_3 \\ \\ \end{array} \qquad \left(\begin{array}{rrrrr} 1 & 0 & -7 & 5 & 2 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 3 & -1 \end{array}\right) \quad \begin{array}{r} 1 \\ 2 \\ \frac{2}{3} \\ 2 \end{array}$$

Step 4 makes each entry above and below the current leading 1 into a 0 by adding suitable multiples of the current row to the other rows.

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 5\mathbf{r}_3 \\ \mathbf{r}_2 \to \mathbf{r}_2 + 2\mathbf{r}_3 \\ \\ \mathbf{r}_4 \to \mathbf{r}_4 - 3\mathbf{r}_3 \end{array} \qquad \left(\begin{array}{rrrrr} 1 & 0 & -7 & 0 & \frac{11}{3} \\ 0 & 1 & 3 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{r} -\frac{7}{3} \\ \frac{10}{3} \\ \frac{2}{3} \\ 0 \end{array}$$

Row 4 now becomes the current row.

$$
\begin{array}{l}
\mathbf{r}_1\\ \mathbf{r}_2\\ \mathbf{r}_3\\ \mathbf{r}_4
\end{array}
\quad
\left(\begin{array}{ccccc|c}
1 & 0 & -7 & 0 & \frac{11}{3} & -\frac{7}{3}\\
0 & 1 & 3 & 0 & -\frac{2}{3} & \frac{10}{3}\\
0 & 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3}\\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

Here, step 1 is not possible so we finish. The matrix is in row-reduced form.

If the strategy has been carried out correctly, then the matrix will be in row-reduced form when we stop, as was the case in the worked exercise above.

In general, when applying the strategy we stop either after every row has been the current row and had the four steps carried out, or when we find that step 1 is not possible, which happens when there are one or more zero rows at the bottom of the matrix.

Exercise C12

Use Strategy C1 to row-reduce the following matrices.

(a) $\begin{pmatrix} 1 & 5 & 1 & 4 & 5 & -1\\ 1 & 5 & 3 & 12 & 11 & 3\\ 3 & 15 & -1 & -4 & 3 & -6\\ -2 & -10 & 1 & 2 & -7 & 6 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & -8 & 8 & -14\\ -1 & 0 & -4 & -6\\ -1 & 8 & -12 & 8\\ 2 & 8 & 0 & 24\\ 1 & 4 & 0 & 14 \end{pmatrix}$

Modifying the strategy

The strategy for row-reducing a matrix works for any matrix and can easily be programmed on a computer. But sometimes when carrying it out by hand we can spot places where carrying out a different row operation will make the calculations easier. Suppose we are working with the matrix below and that we have completed the four steps with row 1 as the current row. Row 2 becomes the current row and step 1 identifies column 2 for the current row's leading 1.

$$
\begin{array}{l}
\mathbf{r}_1\\ \mathbf{r}_2\\ \mathbf{r}_3
\end{array}
\quad
\left(\begin{array}{cccc|c}
1 & 3 & 1 & 2 & 7\\
0 & 4 & 5 & 7 & 16\\
0 & 3 & 4 & 9 & 16
\end{array}\right)
$$

Since the entry is not 0 there is nothing to be done in step 2. In step 3 we are now officially supposed to multiply the current row by $\frac{1}{4}$ in order to create a leading 1, but this will create inconvenient fractions as other entries in row 2. We can, however, spot a different row operation that will also create a leading 1 in the current row, but avoids creating fractions, namely $\mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_3$, since subtracting the 3 from the 4 will create a leading 1. So we perform this alternative row operation as an unofficial version of step 3 and this gives us the matrix below.

$$\mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_3 \qquad \left(\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 7 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 3 & 4 & 9 & 16 \end{array}\right)$$

We now carry out step 4 as normal:

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 3\mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 - 3\mathbf{r}_2 \end{array} \qquad \left(\begin{array}{cccc|c} 1 & 0 & -2 & 8 & 7 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 15 & 16 \end{array}\right)$$

We then carry on with the third row as the current row.

In general, if you are trying to reduce a matrix to row-reduced form, you can use any elementary row operation. Note that even so it can sometimes be impossible to avoid fractions.

Until you are very familiar with row-reducing matrices, it is sensible to follow the systematic strategy very closely, considering modifications only at step 3.

When modifying the strategy and trying to identify an alternative row operation, it is important not to use rows above the current row, as the following exercise illustrates.

Exercise C13

Consider the following matrix where row 2 is the current row.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \qquad \left(\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 7 \\ 0 & 4 & 5 & 7 & 16 \\ 0 & 3 & 4 & 9 & 16 \end{array}\right)$$

Carry out the following row operation and explain why it is not an appropriate alternative operation for step 3.

$$\mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_1$$

When trying to choose an alternative row operation, rows below the current row can be used because the zeros at the beginning of these rows prevent them destroying the progress made so far.

Uniqueness

We have seen that there can be different ways to row-reduce a matrix. Whichever way you choose, you will always get the same answer. This is a consequence of the following theorem, which we state without proof.

Theorem C1

Every matrix has a unique row-reduced form.

Putting it all together

We now have all the techniques necessary for using Gauss–Jordan elimination to solve a system of linear equations using augmented matrices; we just need to put them all together as set out in the following strategy.

Strategy C2

To use Gauss–Jordan elimination to solve a given system of linear equations:

1. form the augmented matrix
2. row-reduce the augmented matrix to obtain its row-reduced form
3. solve the simplified system of linear equations.

Worked Exercise C11

Use Strategy C2 to solve the following system of linear equations.

$$\begin{aligned} 3x + 5y - 12z &= 4 \\ x + y \phantom{{}- 12z} &= 2 \\ 2x + 3y - 4z &= 5 \end{aligned}$$

Solution

We first form the augmented matrix, label the rows and include a check column.

The augmented matrix is

$$\begin{array}{lcc} \mathbf{r}_1 & \multirow{3}{*}{$\left(\begin{array}{ccr|c} 3 & 5 & -12 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 3 & -4 & 5 \end{array}\right)$} & 0 \\ \mathbf{r}_2 & & 4 \\ \mathbf{r}_3 & & 6 \end{array}$$

We now row-reduce this augmented matrix. However, to avoid creating awkward fractions, we perform the row operation $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$ instead of $\mathbf{r}_1 \to \frac{1}{3}\mathbf{r}_1$.

$$\begin{array}{lcc} \mathbf{r}_1 \leftrightarrow \mathbf{r}_2 & \multirow{3}{*}{$\left(\begin{array}{ccr|c} 1 & 1 & 0 & 2 \\ 3 & 5 & -12 & 4 \\ 2 & 3 & -4 & 5 \end{array}\right)$} & 4 \\ & & 0 \\ & & 6 \end{array}$$

We now carry on as usual, following the strategy.

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 3\mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 2\mathbf{r}_1 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 2 & -12 & -2 \\ 0 & 1 & -4 & 1 \end{array}\right) \quad \begin{array}{r} 4 \\ -12 \\ -2 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \frac{1}{2}\mathbf{r}_2 \\ \\ \end{array} \quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -6 & -1 \\ 0 & 1 & -4 & 1 \end{array}\right) \quad \begin{array}{r} 4 \\ -6 \\ -2 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_2 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 0 & 6 & 3 \\ 0 & 1 & -6 & -1 \\ 0 & 0 & 2 & 2 \end{array}\right) \quad \begin{array}{r} 10 \\ -6 \\ 4 \end{array}$$

$$\begin{array}{l} \\ \\ \mathbf{r}_3 \to \frac{1}{2}\mathbf{r}_3 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 0 & 6 & 3 \\ 0 & 1 & -6 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \begin{array}{r} 10 \\ -6 \\ 2 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 6\mathbf{r}_3 \\ \mathbf{r}_2 \to \mathbf{r}_2 + 6\mathbf{r}_3 \\ \\ \end{array} \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \begin{array}{r} -2 \\ 6 \\ 2 \end{array}$$

This matrix is in row-reduced form.

The corresponding system of equations is

$$\begin{aligned} x \quad\quad\quad &= -3, \\ y \quad\; &= 5, \\ z &= 1. \end{aligned}$$

Thus the solution is $x = -3$, $y = 5$, $z = 1$.

Exercise C14

Use Strategy C2 to solve the following system of equations.

$$\begin{aligned} x_1 - 4x_2 - 4x_3 + 3x_4 + 6x_5 &= 2 \\ 2x_1 - 5x_2 - 6x_3 + 6x_4 + 9x_5 &= 3 \\ 2x_1 + 4x_2 \qquad\quad + 9x_4 + 2x_5 &= 0 \end{aligned}$$

3 Matrix operations

In this section you will revise matrices and matrix operations such as matrix addition and matrix multiplication. You will also meet a useful operation called *transposition*.

3.1 Matrix arithmetic

Recall that a matrix with m rows and n columns is an $m \times n$ matrix. An $n \times n$ matrix is called a **square** matrix.

In general, we write $\mathbf{A}$ or (a_{ij}) to denote a matrix:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij}).$$

We call the entry in the ith row and jth column of a matrix $\mathbf{A}$ the **(i,j)-entry**, and often denote it by a_{ij}. Although matrices are usually distinguished in print by the use of bold typeface, when you handwrite them you do not need to underline them, unlike letters that represent vectors.

Matrices are closely related to vectors represented in component form. In Unit A1 you performed vector arithmetic on vectors in both $\mathbb{R}^2$ and $\mathbb{R}^3$, writing a vector in component form as a **row vector**. Such a row vector can be regarded, respectively, as a 1×2 matrix or a 1×3 matrix with real entries; the only difference is the lack of commas in the matrix representation. For example, consider the following 1×3 matrix and the corresponding row vector in $\mathbb{R}^3$:

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad (1, 2, 3).$$

A **column vector** is a vector with the components written vertically; such a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ can be regarded as a matrix with real entries that has just a single column. For example, the following represents both a column vector in $\mathbb{R}^3$ and the corresponding 3×1 matrix:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

It should be clear from the context whether this object is a column vector, with a geometrical interpretation in $\mathbb{R}^3$, or a matrix with real entries.

A matrix may have any size, $m \times n$ for any natural numbers m and n, although we usually write a 1×1 matrix without the brackets and identify it with its single entry.

In this way, matrices can be regarded as a generalisation of vectors with *equality*, the *zero matrix* and the operations of *addition* and *scalar multiplication* defined similarly. Whereas for vectors we defined these in terms of the components, for matrices we define them in terms of the entries. The details are given in the box below.

Matrix arithmetic

Equality Two $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are **equal** if all their corresponding entries agree. We write $\mathbf{A} = \mathbf{B}$.

Zero matrix The $m \times n$ **zero matrix** $\mathbf{0}_{m,n}$ is the $m \times n$ matrix in which all entries are 0. It is denoted by $\mathbf{0}$ when it is clear from the context which size is intended.

Addition The **sum** of two $m \times n$ matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ is the $m \times n$ matrix $\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})$ obtained by adding the corresponding entries:

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix}.$$

Addition of matrices of different sizes is not defined.

Negatives The **negative** of an $m \times n$ matrix $\mathbf{A} = (a_{ij})$ is the $m \times n$ matrix obtained by taking the negatives of its entries:

$$-\mathbf{A} = (-a_{ij}).$$

Subtraction The **difference** of two $m \times n$ matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ is the $m \times n$ matrix $\mathbf{A} - \mathbf{B} = (a_{ij} - b_{ij})$ obtained by subtracting the corresponding entries:

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \cdots & a_{1n} - b_{1n} \\ a_{21} - b_{21} & a_{22} - b_{22} & \cdots & a_{2n} - b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} - b_{m1} & a_{m2} - b_{m2} & \cdots & a_{mn} - b_{mn} \end{pmatrix}.$$

Subtraction of matrices of different sizes is not defined.

Scalar multiplication The **scalar multiple** of an $m \times n$ matrix $\mathbf{A} = (a_{ij})$ by a real number k is the $m \times n$ matrix obtained by multiplying each entry in turn by k.

$$k\mathbf{A} = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix} = (ka_{ij}).$$

For example, consider the matrices below. The first pair are not equal because a pair of corresponding entries differ, and the second pair are not equal as they have different sizes:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 1 \\ 4 & 5 & 6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The following are all examples of zero matrices:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The next worked exercise illustrates matrix addition. In the subsequent exercises you are asked to evaluate the results of various matrix operations.

Worked Exercise C12

Evaluate the following matrix sums, where possible.

(a) $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$

Solution

(a) $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$

(b) This sum is undefined since the matrices are of different sizes.

Exercise C15

Evaluate the following matrix sums, where possible.

(a) $\begin{pmatrix} 1 & -3 \\ -2 & 54 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -3 \\ -2 & 54 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 4 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & 4 & 5 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 6 & -2 \\ 1 & 8 & 2 \\ 0 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 9 \\ 1 & 0 & 4 \\ 3 & -4 & 1 \end{pmatrix}$

Exercise C16

Evaluate the following matrix differences, where possible.

(a) $\begin{pmatrix} 3 & 0 \\ 2 & 7 \end{pmatrix} - \begin{pmatrix} 10 & 3 \\ 1 & 5 \\ 15 & 12 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & 8 & 12 \\ 7 & 2 & -1 \end{pmatrix} - \begin{pmatrix} 3 & 10 & 2 \\ 4 & 9 & 21 \end{pmatrix}$

Exercise C17

Let

$$\mathbf{A} = \begin{pmatrix} 5 & -3 \\ 2 & 3 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 1 \\ -2 & -7 \\ 3 & 5 \end{pmatrix}.$$

Evaluate the following.

(a) $4\mathbf{A}$ (b) $4\mathbf{B}$ (c) $4\mathbf{A} + 4\mathbf{B}$ (d) $4(\mathbf{A} + \mathbf{B})$

In Exercise C15 you should have found that parts (a) and (b) gave the same answer; this is because matrix addition is *commutative*. In fact, matrix addition has the same properties as the additive properties of the real numbers, $\mathbb{R}$, given in Unit A2 *Number systems*. Before listing these properties, we need the following notation:

$M_{m,n}$ denotes the set of all $m \times n$ matrices with real entries.

We can now talk about *arithmetic in* $M_{m,n}$ and the properties it satisfies.

Addition in $M_{m,n}$

A1 Closure For all $\mathbf{A}, \mathbf{B} \in M_{m,n}$,

$$\mathbf{A} + \mathbf{B} \in M_{m,n}.$$

A2 Associativity For all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in M_{m,n}$,

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}).$$

A3 Additive identity For all $\mathbf{A} \in M_{m,n}$ and $\mathbf{0} \in M_{m,n}$,

$$\mathbf{A} + \mathbf{0} = \mathbf{A} = \mathbf{0} + \mathbf{A}.$$

A4 Additive inverses For each $\mathbf{A} \in M_{m,n}$, there is a matrix $-\mathbf{A} \in M_{m,n}$ such that

$$\mathbf{A} + (-\mathbf{A}) = \mathbf{0} = -\mathbf{A} + \mathbf{A}.$$

A5 Commutativity For all $\mathbf{A}, \mathbf{B} \in M_{m,n}$,

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}.$$

The matrix $\mathbf{0}$ is known as the **additive identity** in $M_{m,n}$, and the matrix $-\mathbf{A}$ in property A4 is known as the **additive inverse** of $\mathbf{A}$.

These properties follow from the definition of matrix addition and the corresponding properties of the reals. The next worked exercise proves the closure property (A1) and the commutative property (A5); you are asked to prove the remaining properties in the following exercise.

Worked Exercise C13

By using the corresponding properties for the reals, prove that the following properties hold for $M_{m,n}$ under addition.

(a) The closure property (A1): $\mathbf{A} + \mathbf{B} \in M_{m,n}$.

(b) The commutative property (A5): $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

Solution

Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$.

(a) To prove closure, we add the two matrices and check that the result is another $m \times n$ matrix with real entries.

From the definition of matrix addition, the sum $\mathbf{A} + \mathbf{B}$ is the $m \times n$ matrix with entries $(a_{ij} + b_{ij})$ obtained by adding the corresponding entries.

Since a_{ij} and b_{ij} are real numbers, $a_{ij} + b_{ij} \in \mathbb{R}$. Hence $\mathbf{A} + \mathbf{B} \in M_{m,n}$.

(b) To prove the commutative property, we add the corresponding entries of these two matrices.

The (i, j)-entry of the matrix $\mathbf{A} + \mathbf{B}$ is $a_{ij} + b_{ij}$, and that of $\mathbf{B} + \mathbf{A}$ is $b_{ij} + a_{ij}$.

Since a_{ij} and b_{ij} are real numbers, $a_{ij} + b_{ij} = b_{ij} + a_{ij}$. Thus

$$\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij}) = (b_{ij} + a_{ij}) = \mathbf{B} + \mathbf{A}.$$

Exercise C18

By using the corresponding properties for the reals, prove that the following properties hold for $M_{m,n}$ under addition.

(a) The associative property (A2): $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$.

(b) The identity property (A3): $\mathbf{A} + \mathbf{0} = \mathbf{A} = \mathbf{0} + \mathbf{A}$.

(c) The inverses property (A4): $\mathbf{A} + (-\mathbf{A}) = \mathbf{0} = -\mathbf{A} + \mathbf{A}$.

Recall from Subsection 3.1 of Unit B1 *Symmetry and groups* that a set with a binary operation is a **group** if the following four axioms hold:

G1 (closure), G2 (associativity), G3 (identity) and G4 (inverses).

The first four properties (A1–A4) of matrix addition show that the set of all $m \times n$ matrices with real entries satisfies these four properties; that is, $(M_{m,n}, +)$ is a group with additive identity the zero matrix $\mathbf{0}$, and $-\mathbf{A}$ the additive inverse of $\mathbf{A}$. The final property (A5) shows that it is in fact an abelian group.

Although this unit concentrates on $M_{m,n}$, the set of matrices with real entries, some other sets of $m \times n$ matrices also form a group under addition. For example, the set of $m \times n$ matrices with entries in $\mathbb{Z}$, and those with entries in $\mathbb{C}$, both form a group under addition. However, the set of $m \times n$ matrices with entries in $\mathbb{N}$ does not form a group under addition, since this set of matrices contains neither the zero matrix $\mathbf{0}$, nor the additive inverse $-\mathbf{A}$ of a matrix $\mathbf{A}$ in the set.

Finally in this subsection we return to scalar multiplication of matrices. Recall, from Unit A2, that the reals satisfy a *distributive property* (D1) combining addition and multiplication:

$$a \times (b + c) = (a \times b) + (a \times c), \quad \text{for all } a, b, c \in \mathbb{R}.$$

It turns out that the corresponding property holds for addition and scalar multiplication of matrices; you saw one example of this in Exercise C17(c) and (d) where $4(\mathbf{A} + \mathbf{B})$ and $4\mathbf{A} + 4\mathbf{B}$ were equal.

Combining addition and scalar multiplication in $M_{m,n}$

D1 Distributivity For all $\mathbf{A}, \mathbf{B} \in M_{m,n}$ and $k \in \mathbb{R}$,

$$k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}.$$

You are asked to prove that this property holds in the next exercise.

Exercise C19

By using the corresponding property for the reals, prove that the distributive property (D1) holds for $M_{m,n}$:

$$k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}.$$

3.2 Matrix multiplication

In the previous subsection you saw that matrix addition and scalar multiplication can be defined in terms of matrix entries, much like the corresponding operations for vectors. In this subsection you will revise *matrix multiplication*, which can also be defined in terms of matrix entries, much like the corresponding operation for vectors – the *scalar product*.

Recall from Unit A1 that the scalar product of two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Matrix multiplication is a generalisation of this idea.

To form the product of two matrices $\mathbf{A}$ and $\mathbf{B}$, we combine the rows of $\mathbf{A}$ with the columns of $\mathbf{B}$. The (i, j)-entry of the product $\mathbf{AB}$ is obtained by multiplying each entry in the ith row of $\mathbf{A}$ by the corresponding entry in the jth column of $\mathbf{B}$ and adding the results.

This product is only possible if the number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$.

For example, let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

The number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$, so it is possible to find the product $\mathbf{AB}$.

To obtain the $(1,1)$-entry of the product $\mathbf{AB}$ we combine the *first* row of $\mathbf{A}$ with the *first* column of $\mathbf{B}$:

$$(1 \times 1) + (2 \times 4) + (3 \times 7) = 1 + 8 + 21 = 30.$$

Next, to obtain the $(1,2)$-entry of the product $\mathbf{AB}$, we combine the *first* row of $\mathbf{A}$ with the *second* column of $\mathbf{B}$:

$$(1 \times 2) + (2 \times 5) + (3 \times 8) = 2 + 10 + 24 = 36.$$

Then to obtain the $(1,3)$-entry of the product $\mathbf{AB}$, we combine the *first* row of $\mathbf{A}$ with the *third* column of $\mathbf{B}$:

$$(1 \times 3) + (2 \times 6) + (3 \times 9) = 3 + 12 + 27 = 42.$$

To obtain the entries in the *second* row of the product $\mathbf{AB}$, we combine the *second* row of $\mathbf{A}$ with each of the columns of $\mathbf{B}$ in turn.

In the end we obtain 2×3 entries in the product $\mathbf{AB}$; this matrix has 2 rows and 3 columns as follows.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 30 & 36 & 42 \\ 66 & 81 & 96 \end{pmatrix}$$

One way to remember how to multiply matrices $\mathbf{A}$ and $\mathbf{B}$ is to picture running along the rows of $\mathbf{A}$ and then diving down the columns of $\mathbf{B}$. The example pictured in Figure 13 gives the $(1,2)$-entry.

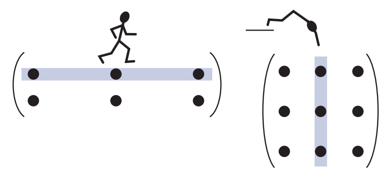


Figure 13 Running along and diving in

Definition

The **product** of an $m \times n$ matrix $\mathbf{A}$ with an $n \times p$ matrix $\mathbf{B}$ is the $m \times p$ matrix $\mathbf{AB}$ whose (i,j)-entry is obtained by multiplying each entry in the ith row of $\mathbf{A}$ by the corresponding entry in the jth column of $\mathbf{B}$ and adding the results.

In symbols, if $\mathbf{C} = \mathbf{AB}$, then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

The product $\mathbf{AB}$ is not defined when the number of columns of the matrix $\mathbf{A}$ is not equal to the number of rows of the matrix $\mathbf{B}$.

Schematically, this can be shown as follows.

$$
\underset{\text{row } i}{m\updownarrow\overset{\longleftarrow n \longrightarrow}{\begin{pmatrix} & & \\ \boxed{* \ \cdots \ *} \\ & & \end{pmatrix}}}
\quad
\underset{\text{column } j}{n\updownarrow\overset{\longleftarrow p \longrightarrow}{\begin{pmatrix} & * & \\ & \vdots & \\ & * & \end{pmatrix}}}
\quad = \quad
\underset{(i,j)\text{-entry}}{m\updownarrow\overset{\longleftarrow p \longrightarrow}{\begin{pmatrix} & & \\ & \boxed{*} & \\ & & \end{pmatrix}}}
$$

Worked Exercise C14

Evaluate (where possible) the matrix products **AB**, where:

(a) $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 1 & 4 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & -2 \end{pmatrix}$.

Solution

(a) The matrix **A** has 2 columns and the matrix **B** has 2 rows, so the product **AB** can be formed.

The product of a 2×2 matrix with a 2×3 one is a 2×3 matrix.

When evaluating a product of matrices, it is advisable to find the entries systematically, either row by row, or column by column. Here, we find the entries row by row.

To find the $(1,1)$-entry of **AB**, we multiply each entry in the *first* row of **A** by the corresponding entry in the *first* column of **B**:

$$(2 \times 3) + (1 \times 1) = 7.$$

Next, to find the $(1,2)$-entry of **AB**, we apply the same procedure to the *first* row of **A** and the *second* column of **B**:

$$(2 \times (-2)) + (1 \times 1) = -3.$$

Next, to find the $(1,3)$-entry of **AB**, we apply the same procedure to the *first* row of **A** and the *third* column of **B**:

$$(2 \times 0) + (1 \times 4) = 4.$$

Together, these give the first row of **AB**:

$$\begin{pmatrix} 7 & -3 & 4 \\ * & * & * \end{pmatrix}$$

We continue by finding the $(2,1)$-entry of **AB** then the $(2,2)$-entry and finally the $(2,3)$-entry, by applying the same procedure to the *second* row of **A** with the columns of **B** in turn. This gives the second row of the product **AB**:

$$\begin{pmatrix} 7 & -3 & 4 \\ -9 & 6 & 0 \end{pmatrix}$$

We have

$$\begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 3 & -2 & 0 \\ 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 7 & -3 & 4 \\ -9 & 6 & 0 \end{pmatrix}.$$

(b) This product is not defined, because the matrix $\mathbf{A}$ has 2 columns and the matrix $\mathbf{B}$ has 1 row.

Exercise C20

Evaluate the following matrix products, where possible.

(a) $\begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 3 & 0 & -4 \end{pmatrix}$ (e) $\begin{pmatrix} 3 & 1 & 2 \\ 0 & 5 & 1 \end{pmatrix}\begin{pmatrix} -2 & 0 & 1 \\ 1 & 3 & 0 \\ 4 & 1 & -1 \end{pmatrix}$

In the previous subsection you saw that addition on the set $M_{m,n}$ of $m \times n$ matrices satisfies the usual properties (A1–A5) for addition. For multiplication of matrices things are not so straightforward. To start with, if $m \neq n$ then the product of two matrices in the set $M_{m,n}$ is not even defined.

So when we consider properties of matrix multiplication we are only interested in products that can be formed. For example, we can say that matrix multiplication is associative because, *whenever these products can be formed*, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$. You will prove this result in Unit C3.

In the next exercise you are asked to prove that matrix multiplication is not commutative.

Exercise C21

(a) Prove that the products $\mathbf{AB}$ and $\mathbf{BA}$ are the same size if and only if $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size.

(b) Prove that matrix multiplication of square matrices of the same size is not commutative by giving a counterexample; that is, find two 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{AB} \neq \mathbf{BA}$.

The fact that matrix multiplication is not commutative means that it is important to describe a matrix product carefully. We say that $\mathbf{AB}$ is the matrix $\mathbf{A}$ *multiplied on the right* by the matrix $\mathbf{B}$, or the matrix $\mathbf{B}$ *multiplied on the left* by the matrix $\mathbf{A}$.

You have seen that the distributive property (D1) holds for multiplication of a matrix by a scalar. Matrix multiplication is also distributive because, *whenever these products can be formed*, $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$. The proof of this is not hard, but it is not very illuminating, so is not given here.

Diagonal and triangular matrices

The entries of a square matrix from the top left-hand corner to the bottom right-hand corner are the **diagonal** entries; the diagonal entries form the **main diagonal** of the matrix. In some texts the main diagonal is called the *leading* or *principal diagonal*. For a square matrix $\mathbf{A} = (a_{ij})$ of size $n \times n$, the diagonal entries are

$$a_{11},\ a_{22},\ \ldots,\ a_{nn}.$$

A matrix that has its only non-zero entries on the main diagonal can be useful.

Definition

A **diagonal matrix** is a square matrix each of whose non-diagonal entries is zero.

For example, the following are diagonal matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To see how diagonal matrices multiply, try the following exercise.

Exercise C22

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix}.$$

Evaluate the following products.

(a) $\mathbf{AB}$ (b) $\mathbf{BA}$ (c) $\mathbf{ABC}$

The product of two diagonal matrices is another diagonal matrix, and the ith diagonal entry of the product is the product of the ith diagonal entries of the matrices being multiplied. Multiplication of *diagonal matrices* is therefore commutative.

More generally, positive **powers** of square matrices are defined as expected:

$$\mathbf{A}^2 = \mathbf{AA}, \quad \mathbf{A}^3 = \mathbf{AAA}, \quad \ldots.$$

Therefore, finding powers of diagonal matrices is straightforward and you will see how this fact can be used to find powers of other square matrices in Unit C4 *Eigenvectors*.

A square matrix with each entry *below* the main diagonal equal to zero is called an **upper triangular matrix**. Similarly, a square matrix with each entry *above* the main diagonal equal to zero is called a **lower triangular matrix**. A square row-reduced matrix is an upper triangular matrix. A square matrix that is both upper triangular and lower triangular is necessarily a diagonal matrix.

Exercise C23

State which of the following matrices are diagonal, upper triangular or lower triangular.

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 9 & 0 \\ 0 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

Identity matrix

You have seen that there are matrices corresponding to the number 0, which is the additive identity in the reals. These matrices are the zero matrices $\mathbf{0}_{m,n}$, each of which is the additive identity in $M_{m,n}$. There are also matrices corresponding to the number 1, which is the multiplicative identity in the reals. These matrices are square matrices called the *identity matrices*, denoted by $\mathbf{I}_n$. The subscript n indicates that the matrix is an $n \times n$ matrix; however, as with the zero matrix, the identity matrix is written simply as $\mathbf{I}$ when the size is clear from the context.

Definition

The **identity matrix $\mathbf{I}_n$** is the $n \times n$ matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Each of the entries is 0 except those on the main diagonal, which are all 1.

For example, the identity matrices $\mathbf{I}_2$, $\mathbf{I}_3$ and $\mathbf{I}_4$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we multiply a 3×2 matrix on the left by $\mathbf{I}_3$ we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}.$$

If we multiply the same 3×2 matrix on the right by $\mathbf{I}_2$ we obtain

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}.$$

Here, a, b, c, d, e and f are any real numbers. In both cases, the matrix is unchanged.

Theorem C2

Let $\mathbf{A}$ be an $m \times n$ matrix. Then

$$\mathbf{I}_m\mathbf{A} = \mathbf{A} = \mathbf{A}\mathbf{I}_n.$$

You are asked to prove this theorem in the next exercise.

Exercise C24

Let $\mathbf{A} = (a_{ij})$ be an $m \times n$ matrix. Prove Theorem C2; that is, prove that $\mathbf{I}_m\mathbf{A} = \mathbf{A}$ and $\mathbf{A}\mathbf{I}_n = \mathbf{A}$.

Hint: Notice that the entries in the ith row of $\mathbf{I}_m$ are all 0 except the entry in the ith position, which is 1.

3.3 Transposition of matrices

There is a simple operation that we can perform on matrices. This operation, called *transposition* or *taking the transpose*, entails interchanging the rows with the columns of the matrix. Thus the transpose of the matrix $\mathbf{A}$, denoted by $\mathbf{A}^T$, has the rows of $\mathbf{A}$ as its columns, taken in the same order. For example,

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 7 \\ -6 & 1 \\ 0 & 4 \end{pmatrix}^T = \begin{pmatrix} 2 & -6 & 0 \\ 7 & 1 & 4 \end{pmatrix}.$$

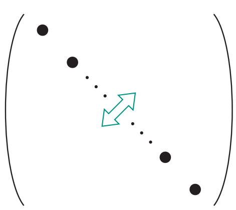


Figure 14 Transposition as reflecting in the main diagonal

Transposition of a *square* matrix can be thought of as reflecting the matrix in the main diagonal, as illustrated in Figure 14.

Definition

The **transpose** of an $m \times n$ matrix $\mathbf{A}$ is the $n \times m$ matrix $\mathbf{A}^T$ whose (i, j)-entry is the (j, i)-entry of $\mathbf{A}$.

Exercise C25

Write down the transpose of each of the following matrices.

(a) $\begin{pmatrix} 1 & 4 \\ 0 & 2 \\ -6 & 10 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 & 2 \\ 0 & 3 & -5 \\ 4 & 7 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 10 & 4 & 6 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

The identity matrix $\mathbf{I}$ is not changed by taking the transpose; that is, $\mathbf{I}^T = \mathbf{I}$. In fact, $\mathbf{A}^T = \mathbf{A}$ for all *diagonal* matrices; you saw one such example in Exercise C25(d).

The operation of transposition has some other useful properties as you will now see.

The rows of the matrix $\mathbf{A}$ form the columns of the matrix $\mathbf{A}^T$, and the columns of $\mathbf{A}^T$ form the rows of $(\mathbf{A}^T)^T$. Therefore the rows of $\mathbf{A}$ form the rows of $(\mathbf{A}^T)^T$; that is, these two matrices are equal:

$$(\mathbf{A}^T)^T = \mathbf{A}.$$

Exercise C26

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

(a) Find $\mathbf{A}^T$, $\mathbf{B}^T$ and $(\mathbf{A}+\mathbf{B})^T$, and verify that $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

(b) Find $\mathbf{C}^T$ and $(\mathbf{AC})^T$, and find an equation relating $(\mathbf{AC})^T$, $\mathbf{A}^T$ and $\mathbf{C}^T$.

The relationships satisfied by the matrices in Exercise C26 hold in general.

Properties of matrix transposition

Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$ matrices. Then:

1. $(\mathbf{A}^T)^T = \mathbf{A}$
2. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{B}$ an $n \times p$ matrix. Then

3. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Symmetric matrices

Some square matrices remain unchanged when transposed. These matrices are called *symmetric* matrices, since they are symmetrical about the main diagonal.

> **Definition**
>
> A square matrix $\mathbf{A}$ is **symmetric** if $\mathbf{A}^T = \mathbf{A}$.

Since $\mathbf{A}^T = \mathbf{A}$ for all diagonal matrices, all diagonal matrices are symmetric. Here are other examples of symmetric matrices:

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ 3 & 6 & 8 & 9 \\ 4 & 7 & 9 & 10 \end{pmatrix}, \quad \begin{pmatrix} -5 & 2 \\ 2 & 3 \end{pmatrix}.$$

3.4 Matrix form of a system of linear equations

In this subsection you will see how a system of linear equations can be expressed in *matrix form* as a product of matrices. This contrasts with the augmented matrices you met in Subsection 2.1, which are an abbreviated notation for the system and involve no products of matrices.

Consider the following system of linear equations.

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 6 \\ x_2 + x_3 &= 1 \\ x_1 + 3x_2 + 5x_3 &= 10 \end{aligned}$$

We can write this system as a matrix equation:

$$\begin{pmatrix} x_1 + 2x_2 + 4x_3 \\ x_2 + x_3 \\ x_1 + 3x_2 + 5x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 10 \end{pmatrix}.$$

Now the 3×1 matrix on the left can be expressed as the product of two matrices, namely the 3×3 matrix of the coefficients and the 3×1 matrix of the unknowns:

$$\begin{pmatrix} x_1 + 2x_2 + 4x_3 \\ x_2 + x_3 \\ x_1 + 3x_2 + 5x_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Thus we have the matrix equation

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 10 \end{pmatrix}.$$

Similarly, we can express any system of linear equations

$$\begin{aligned} a_{11}x_1 + \ a_{12}x_2 + \cdots + \ a_{1n}x_n &= b_1, \\ a_{21}x_1 + \ a_{22}x_2 + \cdots + \ a_{2n}x_n &= b_2, \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

as a matrix product. Let the matrix of coefficients be $\mathbf{A}$, the **coefficient matrix** of the system, that is,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Let the matrix of unknowns be $\mathbf{x}$, and let the matrix of constant terms be $\mathbf{b}$, so

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

The system can then be expressed in **matrix form** as

$$\mathbf{Ax} = \mathbf{b},$$

or in full as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Writing a system of linear equations in matrix form will allow us to manipulate the system using matrix multiplication.

4 Matrix inverses

In this section you will investigate the multiplicative properties of *square* matrices and the existence of multiplicative inverses.

4.1 Matrix inverses

In Section 3 you saw that matrix addition in $M_{m,n}$ satisfies the usual properties (A1–A5) for addition, but things are not so straightforward for multiplication of matrices.

If we restrict our attention to the set $M_{n,n}$ of *square* matrices with real entries, then products of these matrices can always be formed, and so the following properties hold.

Multiplication in $M_{n,n}$

M1 Closure For all $\mathbf{A}, \mathbf{B} \in M_{n,n}$,

$$\mathbf{AB} \in M_{n,n}.$$

M2 Associativity For all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in M_{n,n}$,

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}).$$

M3 Multiplicative identity For all $\mathbf{A} \in M_{n,n}$,

$$\mathbf{AI}_n = \mathbf{A} = \mathbf{I}_n\mathbf{A}.$$

The closure property (M1) follows from the definition of matrix multiplication and the associative property (M2) will be proved in Unit C3. The multiplicative identity property (M3) holds by Theorem C2 and we say that $\mathbf{I}_n$ is the **multiplicative identity** in $M_{n,n}$.

You saw that matrix multiplication is not commutative, even for square matrices, and so the commutative property (M5) does not hold for matrix multiplication in $M_{n,n}$. The distributive property (D1) does hold for matrix addition and matrix multiplication in $M_{n,n}$; that is, matrix multiplication is *distributive* over matrix addition. However, because matrix multiplication is not commutative we have to consider multiplying on the right and left separately.

Combining addition and multiplication in $M_{n,n}$

D1 Distributivity For all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in M_{n,n}$,

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC},$$

and

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}.$$

You may have noticed that one other property is missing from the list of multiplicative properties, namely the multiplicative inverses property (M4).

Recall, from Exercise C21(a), that the products **AB** and **BA** are the same size if and only if **A** and **B** are square matrices of the same size.

We say that **B** is a **multiplicative inverse** of **A** in $M_{n,n}$ if $\mathbf{A}, \mathbf{B} \in M_{n,n}$ and $\mathbf{AB} = \mathbf{I}_n = \mathbf{BA}$. In fact, because the additive inverse of a matrix is usually called the *negative* of the matrix, the multiplicative inverse is usually called the *inverse* of a matrix, where the context is clear.

We now investigate the existence of multiplicative inverses.

Many square matrices do have multiplicative inverses, for example,

$$\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \text{ is an inverse of } \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$$

since

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Similarly,

$$\begin{pmatrix} -1 & -5 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 1 \end{pmatrix} \text{ is an inverse of } \begin{pmatrix} 1 & 3 & -1 \\ -2 & -5 & 1 \\ 4 & 11 & -2 \end{pmatrix}$$

since

$$\begin{pmatrix} 1 & 3 & -1 \\ -2 & -5 & 1 \\ 4 & 11 & -2 \end{pmatrix} \begin{pmatrix} -1 & -5 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} -1 & -5 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & -1 \\ -2 & -5 & 1 \\ 4 & 11 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Just as a real number has at most one multiplicative inverse, or reciprocal, a square matrix has at most one inverse, as we now prove.

Theorem C3

If a square matrix has an inverse, then this inverse is unique.

Proof Let $\mathbf{A}$ be a square matrix, and suppose that $\mathbf{B}$ and $\mathbf{C}$ are both inverses of $\mathbf{A}$. Then $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$ and $\mathbf{AC} = \mathbf{I} = \mathbf{CA}$.

We consider the product $\mathbf{CAB} = \mathbf{C}(\mathbf{AB}) = (\mathbf{CA})\mathbf{B}$.

Multiplying the equation $\mathbf{AB} = \mathbf{I}$ on the left by $\mathbf{C}$, we have

$$\mathbf{C}(\mathbf{AB}) = \mathbf{CI} = \mathbf{C},$$

while multiplying the equation $\mathbf{CA} = \mathbf{I}$ on the right by $\mathbf{B}$ gives

$$(\mathbf{CA})\mathbf{B} = \mathbf{IB} = \mathbf{B}.$$

Since matrix multiplication is associative, it follows that $\mathbf{B} = \mathbf{C}$. ■

Certainly a square zero matrix has no inverse (just as the real number 0 has no reciprocal), since if $\mathbf{0}$ is a square zero matrix, then any product of $\mathbf{0}$ and another matrix is a zero matrix, and so there is no matrix $\mathbf{B}$ such that $\mathbf{0B} = \mathbf{I}$. However, it is natural to ask whether or not every *non-zero* square matrix has an inverse. The next exercise demonstrates that the answer to this question is *no*: it gives an example of a non-zero square matrix with no inverse.

Exercise C27

Let $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

Prove that there is no matrix $\mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\mathbf{AB} = \mathbf{I}$.

In fact, there are many non-zero square matrices with no inverse. The next theorem gives an infinite class of such matrices.

Theorem C4

A square matrix with a zero row has no inverse.

Proof Let $\mathbf{A}$ be a square matrix, one of whose rows, say row i, is a zero row. Then if $\mathbf{B}$ is any matrix of the same size as $\mathbf{A}$, the (i, i)-entry of $\mathbf{AB}$ is 0, since it is obtained by multiplying each entry in row i of $\mathbf{A}$ (a zero row) by the corresponding entry in column i of $\mathbf{B}$. But the (i, i)-entry of $\mathbf{I}$ is 1, which shows that there is no matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I}$. Hence $\mathbf{A}$ has no inverse. ■

Definition

A square matrix that has an inverse is called **invertible**.

An invertible matrix is necessarily a square matrix in order for the products $\mathbf{AB}$ and $\mathbf{BA}$ to exist and be equal.

Since we know by Theorem C3 that if a matrix has an inverse, then this inverse is unique, we denote the unique inverse of an invertible matrix $\mathbf{A}$ by $\mathbf{A}^{-1}$. Thus, for any invertible matrix $\mathbf{A}$,

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}.$$

Both $\mathbf{A}$ and $\mathbf{A}^{-1}$ are square matrices of the same size. It follows from these equations that if $\mathbf{A}$ is an invertible matrix, then $\mathbf{A}^{-1}$ is also invertible, with inverse $\mathbf{A}$; that is,

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}.$$

In other words, the matrices $\mathbf{A}$ and $\mathbf{A}^{-1}$ are *inverses of each other*.

The next worked exercise and the following exercises give some other useful facts about inverses of matrices.

Worked Exercise C15

Let $\mathbf{A}$ be an invertible matrix. Prove that $\mathbf{A}^T$ is invertible, and that $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$.

Solution

The *transpose* of a matrix is the matrix with the rows and columns interchanged.

To prove that $\mathbf{A}^T$ is invertible, with inverse $(\mathbf{A}^{-1})^T$, we have to show that

$$\mathbf{A}^T(\mathbf{A}^{-1})^T = \mathbf{I} = (\mathbf{A}^{-1})^T\mathbf{A}^T.$$

Recall that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

We have

$$\mathbf{A}^T(\mathbf{A}^{-1})^T = (\mathbf{A}^{-1}\mathbf{A})^T = \mathbf{I}^T = \mathbf{I},$$

and, similarly,

$$(\mathbf{A}^{-1})^T\mathbf{A}^T = (\mathbf{A}\mathbf{A}^{-1})^T = \mathbf{I}^T = \mathbf{I}.$$

Therefore $\mathbf{A}^T$ is invertible with inverse $(\mathbf{A}^{-1})^T$.

Exercise C28

Prove that the identity matrix $\mathbf{I}$ is invertible, and that $\mathbf{I}^{-1} = \mathbf{I}$.

Exercise C29

Let $\mathbf{A}$ and $\mathbf{B}$ be invertible matrices of the same size. Prove that $\mathbf{AB}$ is invertible, and that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Notice the reversal of the order of the matrices in the identity

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

This result of Exercise C29 extends to products of any number of matrices; it can be proved using this result and mathematical induction.

Theorem C5

Let $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k$ be invertible matrices of the same size. Then the product $\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_k$ is invertible, with

$$(\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_k)^{-1} = \mathbf{A}_k^{-1}\mathbf{A}_{k-1}^{-1} \cdots \mathbf{A}_1^{-1}.$$

You saw in Subsection 3.1 that $(M_{m,n}, +)$, the set of all $m \times n$ matrices with real entries, forms a group under addition. The results of Exercises C28 and C29, together with the properties M1–M3 for matrix multiplication in $M_{n,n}$, can be used to show that the set of all *invertible* matrices of a particular size and with real entries forms a group under *matrix multiplication*. The restriction of the set to include only invertible matrices is important: without this, the axiom G4 (inverses) clearly fails since, for example, the zero matrix has no inverse.

Theorem C6

The set of all invertible $n \times n$ matrices with real entries forms a group under matrix multiplication.

Proof We check the four group axioms.

G1 Closure Exercise C29 showed that if $\mathbf{A}$ and $\mathbf{B}$ are invertible $n \times n$ matrices then their product $\mathbf{AB}$ is invertible. The product $\mathbf{AB}$ is an $n \times n$ matrix, so group axiom G1 (closure) holds for this set.

G2 Associativity The associative property (M2) holds for matrix multiplication in $M_{n,n}$, so group axiom G2 (associativity) holds.

G3 Identity The identity property (M3) holds for matrix multiplication in $M_{n,n}$, and Exercise C28 shows that $\mathbf{I}_n$ is in the set of all invertible $n \times n$ matrices with real entries. Therefore group axiom G3 (identity) holds.

G4 Inverses The set of all invertible $n \times n$ matrices with real entries is a subset of $M_{n,n}$. By definition every matrix in the set of invertible matrices has an inverse, and this inverse is itself invertible and therefore in the set, so axiom G4 (inverses) holds.

Hence the set of all invertible $n \times n$ matrices with real entries under the operation of matrix multiplication satisfies the four group axioms, and so is a group. ■

4.2 Invertibility Theorem

The following two questions may already have occurred to you as you worked through the previous subsection. First, how can we determine whether or not a given square matrix is invertible? Second, if we know that a matrix is invertible, how can we find its inverse? The next theorem, which we will prove in Subsection 4.5, answers both these questions.

Theorem C7 Invertibility Theorem

(a) A square matrix is invertible if and only if its row-reduced form is $\mathbf{I}$.

(b) Any sequence of elementary row operations that transforms a matrix $\mathbf{A}$ to $\mathbf{I}$ also transforms $\mathbf{I}$ to $\mathbf{A}^{-1}$.

To illustrate this theorem, consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 9 \end{pmatrix}.$$

Suppose that we wish to determine whether or not $\mathbf{A}$ is invertible and, if it is, to find $\mathbf{A}^{-1}$.

Below, on the left, we row-reduce $\mathbf{A}$ in the usual way. On the right, we perform the same sequence of elementary row operations on the 2×2 identity matrix.

$$\begin{array}{llll}
\mathbf{r}_1 & \begin{pmatrix} 1 & 3 \\ 2 & 9 \end{pmatrix} & \mathbf{r}_1 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\mathbf{r}_2 & & \mathbf{r}_2 & \\
\mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 & \begin{pmatrix} 1 & 3 \\ 0 & 3 \end{pmatrix} & \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 & \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \\
\mathbf{r}_2 \to \frac{1}{3}\mathbf{r}_2 & \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} & \mathbf{r}_2 \to \frac{1}{3}\mathbf{r}_2 & \begin{pmatrix} 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \\
\mathbf{r}_1 \to \mathbf{r}_1 - 3\mathbf{r}_2 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \mathbf{r}_1 \to \mathbf{r}_1 - 3\mathbf{r}_2 & \begin{pmatrix} 3 & -1 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}
\end{array}$$

The row-reduced form of $\mathbf{A}$ is $\mathbf{I}$, and so we conclude from the first part of the Invertibility Theorem that $\mathbf{A}$ is an invertible matrix.

By the second part of the Invertibility Theorem, the final matrix on the right above must be $\mathbf{A}^{-1}$; that is,

$$\mathbf{A}^{-1} = \begin{pmatrix} 3 & -1 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

You should check that this matrix is indeed the inverse of $\mathbf{A}$.

To apply the Invertibility Theorem to find the inverse of a matrix $\mathbf{A}$, we have to perform the same sequence of elementary row operations on both $\mathbf{A}$ and $\mathbf{I}$. We can do this conveniently in the following way. We begin by writing $\mathbf{A}$ and $\mathbf{I}$ alongside each other, separated by a vertical line, giving a larger matrix, which we may denote by $(\mathbf{A} \mid \mathbf{I})$. We then row-reduce $(\mathbf{A} \mid \mathbf{I})$ in the usual way (with the check column included). When we do this, the above calculation is as follows.

$$\begin{array}{ll}
\begin{matrix}\mathbf{r}_1 \\ \mathbf{r}_2\end{matrix} & \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 9 & 0 & 1 \end{array}\right) \begin{matrix} 5 \\ 12 \end{matrix} \\[2ex]
\mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 & \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 3 & -2 & 1 \end{array}\right) \begin{matrix} 5 \\ 2 \end{matrix} \\[2ex]
\mathbf{r}_2 \to \frac{1}{3}\mathbf{r}_2 & \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} \end{array}\right) \begin{matrix} 5 \\ \frac{2}{3} \end{matrix} \\[2ex]
\mathbf{r}_1 \to \mathbf{r}_1 - 3\mathbf{r}_2 & \left(\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} \end{array}\right) \begin{matrix} 3 \\ \frac{2}{3} \end{matrix}
\end{array}$$

Thus the Invertibility Theorem (Theorem C7) gives us the following strategy.

Strategy C3

To determine whether or not a given square matrix $\mathbf{A}$ is invertible, and to find its inverse if it is, do the following.

Write down $(\mathbf{A} \mid \mathbf{I})$, and row-reduce it until the left half is in row-reduced form.

- If the left half is the identity matrix, then the right half is $\mathbf{A}^{-1}$.
- Otherwise, $\mathbf{A}$ is not invertible.

You may find it helpful to remember the following scheme for this strategy:

$$\begin{gathered}(\mathbf{A} \mid \mathbf{I}) \\ \downarrow \\ (\mathbf{I} \mid \mathbf{A}^{-1}).\end{gathered}$$

Strategy C3 is most useful for matrices of size 3×3 and larger. In Section 5 you will revise a quick method for determining whether or not a 2×2 matrix is invertible, and for writing down its inverse if it is invertible.

Worked Exercise C16

Determine whether or not each of the following matrices is invertible, and find the inverse if it exists.

(a) $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -4 \\ 3 & 2 & 10 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 1 & 7 \\ 2 & 4 & 8 \end{pmatrix}$

Solution

We use Strategy C3, and again add the row-sum check to help pick up any arithmetical errors.

(a) We row-reduce the matrix $(\mathbf{A} \mid \mathbf{I})$.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \quad \left(\begin{array}{rrr|rrr} 1 & 1 & 2 & 1 & 0 & 0 \\ -1 & 0 & -4 & 0 & 1 & 0 \\ 3 & 2 & 10 & 0 & 0 & 1 \end{array}\right) \begin{array}{r} 5 \\ -4 \\ 16 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 + \mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 3\mathbf{r}_1 \end{array} \quad \left(\begin{array}{rrr|rrr} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & -1 & 4 & -3 & 0 & 1 \end{array}\right) \begin{array}{r} 5 \\ 1 \\ 1 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 + \mathbf{r}_2 \end{array} \quad \left(\begin{array}{rrr|rrr} 1 & 0 & 4 & 0 & -1 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 2 & -2 & 1 & 1 \end{array}\right) \begin{array}{r} 4 \\ 1 \\ 2 \end{array}$$

$$\begin{array}{l} \\ \\ \mathbf{r}_3 \to \frac{1}{2}\mathbf{r}_3 \end{array} \quad \left(\begin{array}{rrr|rrr} 1 & 0 & 4 & 0 & -1 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} \end{array}\right) \begin{array}{r} 4 \\ 1 \\ 1 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 4\mathbf{r}_3 \\ \mathbf{r}_2 \to \mathbf{r}_2 + 2\mathbf{r}_3 \\ \\ \end{array} \quad \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 4 & -3 & -2 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} \end{array}\right) \begin{array}{r} 0 \\ 3 \\ 1 \end{array}$$

The left half has been reduced to $\mathbf{I}$, so $\mathbf{A}$ is invertible; its inverse is given by the right half, that is,

$$\mathbf{A}^{-1} = \begin{pmatrix} 4 & -3 & -2 \\ -1 & 2 & 1 \\ -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

(b) We row-reduce $(\mathbf{B} \mid \mathbf{I})$.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \quad \left(\begin{array}{rrr|rrr} 1 & 3 & 5 & 1 & 0 & 0 \\ 3 & 1 & 7 & 0 & 1 & 0 \\ 2 & 4 & 8 & 0 & 0 & 1 \end{array}\right) \begin{array}{r} 10 \\ 12 \\ 15 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 3\mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 2\mathbf{r}_1 \end{array} \quad \left(\begin{array}{rrr|rrr} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & -8 & -8 & -3 & 1 & 0 \\ 0 & -2 & -2 & -2 & 0 & 1 \end{array}\right) \begin{array}{r} 10 \\ -18 \\ -5 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to -\frac{1}{8}\mathbf{r}_2 \\ \\ \end{array} \quad \left(\begin{array}{rrr|rrr} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{3}{8} & -\frac{1}{8} & 0 \\ 0 & -2 & -2 & -2 & 0 & 1 \end{array}\right) \begin{array}{r} 10 \\ \frac{9}{4} \\ -5 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 3\mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 + 2\mathbf{r}_2 \end{array} \quad \left(\begin{array}{rrr|rrr} 1 & 0 & 2 & -\frac{1}{8} & \frac{3}{8} & 0 \\ 0 & 1 & 1 & \frac{3}{8} & -\frac{1}{8} & 0 \\ 0 & 0 & 0 & -\frac{5}{4} & -\frac{1}{4} & 1 \end{array}\right) \begin{array}{r} \frac{13}{4} \\ \frac{9}{4} \\ -\frac{1}{2} \end{array}$$

The left half is now in row-reduced form, but is not the identity matrix. Therefore $\mathbf{B}$ is not invertible.

If, for a general matrix $\mathbf{A}$, it becomes clear while you are row-reducing $(\mathbf{A} \mid \mathbf{I})$ that the left half will not reduce to the identity matrix (for example, if a zero row appears in the left half), then you can stop the row-reduction immediately, and conclude that $\mathbf{A}$ is not invertible. There is no point in continuing until the left half is in row-reduced form.

Exercise C30

Use Strategy C3 to determine whether or not each of the following matrices is invertible, and find the inverse if it exists.

(a) $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 1 & 1 & -4 \\ 2 & 1 & -6 \\ -3 & -1 & 9 \end{pmatrix}$ (c) $\mathbf{C} = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 2 & 4 \\ 5 & 10 & 5 \end{pmatrix}$

4.3 Invertibility and systems of linear equations

We can use matrix inverses to give us another method for solving certain systems of linear equations.

Consider the system that we solved by Gauss–Jordan elimination in Worked Exercise C1.

$$\begin{aligned} 2x + 4y &= 10 \\ 4x + y &= 6 \end{aligned}$$

You saw in Subsection 3.4 that such systems may be expressed in matrix form as

$$\begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 6 \end{pmatrix}.$$

In Exercise C30(a) you found that this coefficient matrix is invertible:

$$\begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{14} & \frac{2}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{pmatrix}.$$

Multiplying both sides of the matrix form of the system on the left by the inverse of the coefficient matrix, we obtain

$$\begin{pmatrix} -\frac{1}{14} & \frac{2}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{14} & \frac{2}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{pmatrix} \begin{pmatrix} 10 \\ 6 \end{pmatrix},$$

that is,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

So the system has the unique solution $x = 1$, $y = 2$.

In general, suppose that $\mathbf{Ax} = \mathbf{b}$ is the matrix form of a system of linear equations, and that the coefficient matrix $\mathbf{A}$ is invertible. Then we can multiply both sides of the equation $\mathbf{Ax} = \mathbf{b}$ on the left by $\mathbf{A}^{-1}$ to yield $\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b}$; that is, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. It seems, then, that the system has the unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

However, we have to be careful before making this claim. Whenever we manipulate an equation in order to solve it, we have to be sure that the manipulation yields a second equation *equivalent* to the first (otherwise the two equations might have different solution sets).

In this case, we have to be sure that

$$\mathbf{Ax} = \mathbf{b} \quad \text{if and only if} \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

We showed above that if $\mathbf{Ax} = \mathbf{b}$, then multiplying both sides on the left by $\mathbf{A}^{-1}$ yields $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$; in other words, we proved that $\mathbf{Ax} = \mathbf{b}$ implies $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. It remains to prove that $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ implies $\mathbf{Ax} = \mathbf{b}$, and fortunately this is just as easy: if $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, then multiplying both sides of this equation on the left by $\mathbf{A}$ yields $\mathbf{Ax} = \mathbf{AA}^{-1}\mathbf{b}$; that is, $\mathbf{Ax} = \mathbf{b}$.

So multiplying both sides of $\mathbf{Ax} = \mathbf{b}$ on the left by $\mathbf{A}^{-1}$ *does* yield an equivalent equation. We have proved the following theorem.

Theorem C8

Let $\mathbf{A}$ be an invertible matrix. Then the system of linear equations $\mathbf{Ax} = \mathbf{b}$ has the unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

Exercise C31

Use Theorem C8 to solve the following system of linear equations.

$$\begin{aligned} x + y + 2z &= 1 \\ -x \phantom{{}+y} - 4z &= 2 \\ 3x + 2y + 10z &= -1 \end{aligned}$$

In Worked Exercise C16 you saw that

$$\begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -4 \\ 3 & 2 & 10 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & -3 & -2 \\ -1 & 2 & 1 \\ -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

In general, it is worth using the method of Theorem C8 only if we have already calculated the inverse of the coefficient matrix. To use the method of Theorem C8 to solve $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is an $n \times n$ invertible matrix, we first invert $\mathbf{A}$. This involves row-reducing the matrix $(\mathbf{A} \mid \mathbf{I})$. We then calculate the matrix product $\mathbf{A}^{-1}\mathbf{b}$. On the other hand, the method of Section 2 using Gauss–Jordan elimination involves only row-reducing the matrix $(\mathbf{A} \mid \mathbf{b})$ and so is usually quicker.

Theorem C8 shows, in particular, that if the coefficient matrix $\mathbf{A}$ of a system of linear equations $\mathbf{Ax} = \mathbf{b}$ is invertible, then the system has a *unique* solution. The converse of this result is also true – we prove this in the next theorem.

This theorem gives some important relationships between the invertibility of a matrix and the number of solutions of a system of linear equations that has this matrix as its coefficient matrix. The theorem states that three conditions are *equivalent*: any one of the conditions implies any other one.

Theorem C9

Let $\mathbf{A}$ be an $n \times n$ matrix. Then the following statements are equivalent.

(a) $\mathbf{A}$ is invertible.

(b) The system $\mathbf{Ax} = \mathbf{b}$ has a unique solution for each $n \times 1$ matrix $\mathbf{b}$.

(c) The system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.

Proof We show that (a) $\implies$ (b), (b) $\implies$ (c) and (c) $\implies$ (a), which shows that the conditions are equivalent.

(a) $\implies$ (b)
Suppose that $\mathbf{A}$ is an invertible $n \times n$ matrix. Then, by Theorem C8, for any $n \times 1$ matrix $\mathbf{b}$, the system $\mathbf{Ax} = \mathbf{b}$ has the unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

(b) $\implies$ (c)
Suppose that the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution for each $n \times 1$ matrix $\mathbf{b}$. Then, in particular, the homogeneous system $\mathbf{Ax} = \mathbf{0}$ has a unique solution. But every homogeneous system has the trivial solution; thus this unique solution must be the trivial one.

(c) $\implies$ (a)
Suppose that the system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution. Then row-reducing the augmented matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right)$$

of the system must yield

$$\left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{array}\right),$$

since this is the row-reduced matrix that corresponds to each unknown being 0. If we now ignore the last column of each of the matrices appearing in this row-reduction, we are left with a reduction of $\mathbf{A}$ to $\mathbf{I}$. Hence, by the Invertibility Theorem (Theorem C7), $\mathbf{A}$ is invertible. ■

4.4 Elementary matrices

In this subsection you will meet a class of square matrices associated with elementary row operations and investigate their properties.

We will use these matrices and their properties in Subsection 4.5 to help prove the Invertibility Theorem (Theorem C7). We will also find them useful later.

Consider the following matrices:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$$

They are obtained by performing, on the 3×3 identity matrix, the elementary row operations $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$, $\mathbf{r}_2 \to 5\mathbf{r}_2$ and $\mathbf{r}_3 \to \mathbf{r}_3 + 2\mathbf{r}_2$, respectively.

Definition

A matrix obtained by performing an elementary row operation on an identity matrix is an **elementary matrix**.

The elementary row operation that is performed to obtain an elementary matrix from an identity matrix is called the elementary row operation *associated with* that elementary matrix.

We now demonstrate the most important property of elementary matrices. Below, we show the effect of multiplying the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

on the left by each of the above elementary matrices. Notice that in each case, the resulting matrix is precisely the matrix that is obtained when the row operation associated with the elementary matrix is performed on $\mathbf{A}$.

$$\underset{\substack{\text{elementary matrix}\\ \text{associated with}\\ \mathbf{r}_1 \leftrightarrow \mathbf{r}_2}}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \quad \underset{\mathbf{A}}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}} = \underset{\substack{\text{matrix obtained when}\\ \mathbf{r}_1 \leftrightarrow \mathbf{r}_2\\ \text{is performed on } \mathbf{A}}}{\begin{pmatrix} 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \\ 9 & 10 & 11 & 12 \end{pmatrix}}$$

$$\underset{\substack{\text{elementary matrix}\\ \text{associated with}\\ \mathbf{r}_2 \to 5\mathbf{r}_2}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \quad \underset{\mathbf{A}}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}} = \underset{\substack{\text{matrix obtained when}\\ \mathbf{r}_2 \to 5\mathbf{r}_2\\ \text{is performed on } \mathbf{A}}}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 25 & 30 & 35 & 40 \\ 9 & 10 & 11 & 12 \end{pmatrix}}$$

$$\underset{\substack{\text{elementary matrix}\\ \text{associated with}\\ \mathbf{r}_3 \to \mathbf{r}_3 + 2\mathbf{r}_2}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}} \quad \underset{\mathbf{A}}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}} = \underset{\substack{\text{matrix obtained when}\\ \mathbf{r}_3 \to \mathbf{r}_3 + 2\mathbf{r}_2\\ \text{is performed on } \mathbf{A}}}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 19 & 22 & 25 & 28 \end{pmatrix}}$$

There is nothing special about the above elementary matrices, or about the above matrix $\mathbf{A}$. In the next exercise you will find that other elementary matrices behave similarly.

Exercise C32

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}$.

(a) Write down the 2×2 elementary matrix associated with the elementary row operation $\mathbf{r}_1 \to 5\mathbf{r}_1$.

Multiply $\mathbf{A}$ on the left by this elementary matrix, and check that the resulting matrix is the same as the matrix obtained when $\mathbf{r}_1 \to 5\mathbf{r}_1$ is performed on $\mathbf{A}$.

(b) Write down the 4×4 elementary matrix associated with the elementary row operation $\mathbf{r}_2 \to \mathbf{r}_2 + 3\mathbf{r}_4$.

Multiply $\mathbf{B}$ on the left by this elementary matrix, and check that the resulting matrix is the same as the matrix obtained when $\mathbf{r}_2 \to \mathbf{r}_2 + 3\mathbf{r}_4$ is performed on $\mathbf{B}$.

Notice that the number of columns of the elementary matrix used must equal the number of rows of the matrix upon which the elementary operation is to be performed; that is, the elementary row operations should be applied to an appropriately sized identity matrix to obtain the elementary matrix required.

In general, we have the following theorem, which we state without proof.

Theorem C10

Let $\mathbf{E}$ be an elementary matrix, and let $\mathbf{A}$ be any matrix with the same number of rows as $\mathbf{E}$. Then the product $\mathbf{EA}$ is the same as the matrix obtained when the elementary row operation associated with $\mathbf{E}$ is performed on $\mathbf{A}$.

Theorem C10 tells us that if we perform an elementary row operation on a matrix $\mathbf{A}$ with m rows, then the resulting matrix is $\mathbf{EA}$, where $\mathbf{E}$ is the $m \times m$ elementary matrix associated with the row operation.

What happens if we perform a *sequence* of k elementary row operations on a matrix $\mathbf{A}$ with m rows? Let $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$ be the $m \times m$ elementary matrices associated with the row operations in the sequence, in the same order. The first row operation is performed on $\mathbf{A}$, producing the matrix $\mathbf{E}_1\mathbf{A}$; the second row operation is then performed on *this* matrix, producing the matrix $\mathbf{E}_2(\mathbf{E}_1\mathbf{A}) = \mathbf{E}_2\mathbf{E}_1\mathbf{A}$; and so on. After the whole sequence of k row operations has been performed, the resulting matrix is $\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A}$. Notice that the order of the elementary matrices in this matrix product is the *reverse* of the order in which their associated row operations are performed.

This fact will be useful later, and we record it as a corollary to Theorem C10.

Corollary C11

Let $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$ be the $m \times m$ elementary matrices associated with a sequence of k elementary row operations carried out on a matrix $\mathbf{A}$ with m rows, in the same order. Then, after the sequence of row operations has been performed, the resulting matrix is

$$\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A}.$$

For example, earlier, to illustrate the Invertibility Theorem (Theorem C7), we performed the sequence of row operations

$$\mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1, \quad \mathbf{r}_2 \to \tfrac{1}{3}\mathbf{r}_2, \quad \mathbf{r}_1 \to \mathbf{r}_1 - 3\mathbf{r}_2,$$

on the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 9 \end{pmatrix}$$

to produce the identity matrix $\mathbf{I}_2$.

By Corollary C11 we have the following, which you should check by evaluating the product on the right-hand side.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 9 \end{pmatrix}.$$

We now explore some other useful connections between elementary row operations and elementary matrices. We begin by introducing a further property of elementary row operations.

In the following example, the second elementary row operation undoes the effect of the first.

$$\begin{array}{ll} \mathbf{r}_1 & \\ \mathbf{r}_2 & \end{array} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\mathbf{r}_2 \to \mathbf{r}_2 + 3\mathbf{r}_1 \quad \begin{pmatrix} 1 & 2 & 3 \\ 7 & 11 & 15 \end{pmatrix}$$

$$\mathbf{r}_2 \to \mathbf{r}_2 - 3\mathbf{r}_1 \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

In fact, given any elementary row operation, it is easy to write down an *inverse* elementary row operation that undoes the effect of the first, as summarised in the following table.

Elementary row operation	Inverse elementary row operation
$\mathbf{r}_i \leftrightarrow \mathbf{r}_j$	$\mathbf{r}_i \leftrightarrow \mathbf{r}_j$
$\mathbf{r}_i \to c\,\mathbf{r}_i \quad (c \neq 0)$	$\mathbf{r}_i \to (1/c)\,\mathbf{r}_i$
$\mathbf{r}_i \to \mathbf{r}_i + c\,\mathbf{r}_j$	$\mathbf{r}_i \to \mathbf{r}_i - c\,\mathbf{r}_j$

Exercise C33

Write down the inverse of each of the following elementary row operations. Check your answer in each case by carrying out the sequence of two row operations on the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

(a) $\mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_2$ (b) $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$ (c) $\mathbf{r}_2 \to -3\mathbf{r}_2$

Note that if two elementary row operations are such that the second is the inverse of the first, then the first is the inverse of the second – so it makes sense to say that they are *inverses of each other*, or that they form an **inverse pair**. For example, the inverse of $\mathbf{r}_2 \to \mathbf{r}_2 + 3\mathbf{r}_1$ is $\mathbf{r}_2 \to \mathbf{r}_2 - 3\mathbf{r}_1$, and the inverse of $\mathbf{r}_2 \to \mathbf{r}_2 - 3\mathbf{r}_1$ is $\mathbf{r}_2 \to \mathbf{r}_2 - (-3)\mathbf{r}_1$, that is, $\mathbf{r}_2 \to \mathbf{r}_2 + 3\mathbf{r}_1$. So $\mathbf{r}_2 \to \mathbf{r}_2 + 3\mathbf{r}_1$ and $\mathbf{r}_2 \to \mathbf{r}_2 - 3\mathbf{r}_1$ are inverses of each other.

Now consider the following pair of 2×2 elementary matrices associated with the inverse pair of elementary row operations $\mathbf{r}_2 \to \mathbf{r}_2 + 3\mathbf{r}_1$ and $\mathbf{r}_2 \to \mathbf{r}_2 - 3\mathbf{r}_1$:

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}.$$

These two matrices are themselves inverses of each other, as we can easily check:

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

This connection between inverse pairs of elementary row operations and inverse pairs of elementary matrices holds in general.

Theorem C12

Let $\mathbf{E}_1$ and $\mathbf{E}_2$ be elementary matrices of the same size whose associated elementary row operations are inverses of each other. Then $\mathbf{E}_1$ and $\mathbf{E}_2$ are inverses of each other.

Proof In this proof we refer to the row operations associated with $\mathbf{E}_1$ and $\mathbf{E}_2$ as row operation 1 and row operation 2, respectively.

By Corollary C11, $\mathbf{E}_2\mathbf{E}_1\mathbf{I}$ is the matrix produced when row operations 1 and 2 are performed, in that order, on $\mathbf{I}$. Similarly, $\mathbf{E}_1\mathbf{E}_2\mathbf{I}$ is the matrix produced when row operations 2 and 1 are performed, in that order, on $\mathbf{I}$. But each of these two row operations undoes the effect of the other, so $\mathbf{E}_2\mathbf{E}_1\mathbf{I} = \mathbf{I}$ and $\mathbf{E}_1\mathbf{E}_2\mathbf{I} = \mathbf{I}$; that is,

$$\mathbf{E}_2\mathbf{E}_1 = \mathbf{I} = \mathbf{E}_1\mathbf{E}_2.$$

Thus $\mathbf{E}_1$ and $\mathbf{E}_2$ are inverses of each other. ■

Theorem C12 has the following corollary.

Corollary C13

Every elementary matrix is invertible, and its inverse is also an elementary matrix.

Proof Let $\mathbf{E}$ be an elementary matrix. Then $\mathbf{E}$ has an associated elementary row operation. This associated elementary row operation has an inverse operation, and the elementary matrix of the same size as $\mathbf{E}$ associated with this inverse operation is the inverse of $\mathbf{E}$, by Theorem C12. ■

Exercise C34

Use the method suggested by the proof of Corollary C13 to find the inverse of the elementary matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4.5 Proof of the Invertibility Theorem

We are now ready to prove the Invertibility Theorem, using elementary matrices and their properties. We first remind you of the theorem.

Theorem C7 Invertibility Theorem

(a) A square matrix is invertible if and only if its row-reduced form is $\mathbf{I}$.

(b) Any sequence of elementary row operations that transforms a matrix $\mathbf{A}$ to $\mathbf{I}$ also transforms $\mathbf{I}$ to $\mathbf{A}^{-1}$.

Proof Let $\mathbf{A}$ be an $n \times n$ matrix, and let the row-reduced form of $\mathbf{A}$ be $\mathbf{U}$. Let $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$ be the $n \times n$ elementary matrices associated with a sequence of k elementary row operations that transforms $\mathbf{A}$ to $\mathbf{U}$. Then, by Corollary C11,

$$\mathbf{U} = \mathbf{BA},$$

where $\mathbf{B} = \mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1$. Now $\mathbf{B}$ is invertible – since every elementary matrix is invertible (by Corollary C13), and a product of invertible matrices is invertible (by Theorem C5).

(a) We start by proving the *only if* statement.

First we show that if $\mathbf{A}$ is invertible, then $\mathbf{U} = \mathbf{I}$.

Suppose that $\mathbf{A}$ is invertible. Then $\mathbf{U}$ is a product of invertible matrices ($\mathbf{B}$ and $\mathbf{A}$); hence $\mathbf{U}$ is invertible.

Therefore $\mathbf{U}$ does not have a zero row (since, by Theorem C4, a square matrix with a zero row is not invertible), and so from the definition of row-reduced form, it has a leading 1 in each of its n rows. Each of these n leading ones lies in a different column; so, since $\mathbf{U}$ has only n columns, each column must contain a leading 1. Thus the leading 1 in the top row must lie in the left-most position, and the leading 1 in each subsequent row must lie just one position to the right of the leading 1 in the row immediately above. All the entries above and below these leading ones are zeros. Hence $\mathbf{U} = \mathbf{I}$.

We now prove the *if* statement.

Next, we show that if $\mathbf{U} = \mathbf{I}$, then $\mathbf{A}$ is invertible.

Suppose that $\mathbf{U} = \mathbf{I}$. Then

$$\mathbf{I} = \mathbf{BA}. \tag{5}$$

Multiplying both sides of equation (5) on the left by $\mathbf{B}^{-1}$ yields

$$\mathbf{B}^{-1}\mathbf{I} = \mathbf{B}^{-1}\mathbf{B}\mathbf{A},$$

that is,

$$\mathbf{B}^{-1} = \mathbf{A}.$$

Multiplying both sides of *this* equation on the right by $\mathbf{B}$ yields

$$\mathbf{B}^{-1}\mathbf{B} = \mathbf{A}\mathbf{B},$$

that is,

$$\mathbf{I} = \mathbf{A}\mathbf{B}. \qquad (6)$$

Equations (5) and (6) together tell us that $\mathbf{A}$ is invertible, and that $\mathbf{A}^{-1} = \mathbf{B}$.

(b) It follows from the proof of part (a) that if $\mathbf{U} = \mathbf{I}$, then $\mathbf{A}$ is invertible and $\mathbf{A}^{-1} = \mathbf{B}$; that is, $\mathbf{A}^{-1} = \mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1$.

This equation can be written as

$$\mathbf{A}^{-1} = \mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{I},$$

which tells us that $\mathbf{A}^{-1}$ is the matrix produced by performing on $\mathbf{I}$ the sequence of row operations associated with $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$. ■

5 Determinants

In this section you will revise the *determinant* of a 2×2 matrix, and see how this concept extends to $n \times n$ matrices.

5.1 Systems of linear equations and determinants

Determinants arise naturally in the study of systems of linear equations.

In 1693 Gottfried Wilhelm Leibniz (1646–1716) wrote a letter to the Marquis de l'Hôpital in which he demonstrated a method for solving a system of three simultaneous equations which involved calculating what we now call the determinant of a 3×3 matrix, and went on to give a general (although rather unclear) rule for calculating the determinant of an $n \times n$ matrix.

The actual term 'determinant' was introduced by Carl Friedrich Gauss (1777–1855) in his *Disquistiones Arithmeticae* of 1801, but it was Augustin-Louis Cauchy (1789–1857) who in 1812, adapting the term from Gauss, first used it in its modern sense and began to develop a proper theory of determinants.

Gabriel Cramer

Colin Maclaurin

This connection between determinants and systems of linear equations was made explicit by Gabriel Cramer in a method known as *Cramer's rule*. If a unique solution exists for a system of n linear equations in n unknowns, then this solution can be found by evaluating determinants. You will see Cramer's rule for a system of two linear equations in two unknowns; it is rather unwieldy to use for larger systems. However, Cramer's rule gives an expression for each unknown individually, so it makes it possible to find one unknown without solving the whole system.

Cramer's rule is named after the Swiss mathematician Gabriel Cramer (1704–1752) who presented it in his *Introduction à l'analyse des lignes courbes algébriques* (*Introduction to the Analysis of Algebraic Curved Lines*) of 1750, although the Scottish mathematician Colin Maclaurin (1698–1746) had already described the rule in his *Treatise of Algebra* (1748) written in 1730 but not published until after his death.

Determinant of a 2×2 matrix

We start by looking at a system of two equations in two unknowns, where the coefficients of the system are real numbers.

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned} \quad \text{or} \quad \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Using Gauss–Jordan elimination, the following solution can be found,

$$x = \frac{c_1b_2 - b_1c_2}{a_1b_2 - b_1a_2}, \quad y = \frac{a_1c_2 - c_1a_2}{a_1b_2 - b_1a_2}, \tag{7}$$

provided that $a_1b_2 - b_1a_2$ is not zero. (You can check this solution by substitution.) We call the expression $a_1b_2 - b_1a_2$ the *determinant* of the coefficient matrix. Each term in this expression contains the letters a and b, and the subscripts 1 and 2, in some order.

The definition we give for the determinant of a 2×2 matrix is in a form that is easier to remember.

Definition

The **determinant** of a 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is

$$\det \mathbf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

You might find it helpful to remember the scheme shown in Figure 15.

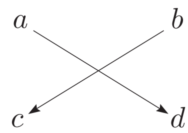


Figure 15 Scheme for 2×2 determinant

We write det $\mathbf{A}$, and use vertical bars '| ... |' around the matrix entries, in place of the round brackets, to denote the determinant. Some texts use the notation $|\mathbf{A}|$ rather than det $\mathbf{A}$.

For example, let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$; then

$$\det \mathbf{A} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1 \times 4) - (2 \times 3) = -2.$$

Exercise C35

Evaluate the determinant of each of the following matrices.

(a) $\begin{pmatrix} 5 & 1 \\ 4 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 10 & -4 \\ -5 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 7 & 3 \\ 17 & 2 \end{pmatrix}$

Notice that the numerators of the solutions for x and y in (7) can also be written as determinants:

$$c_1b_2 - b_1c_2 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \quad \text{and} \quad a_1c_2 - c_1a_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

So we could write these solutions as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

This is Cramer's rule for a system of two linear equations in two unknowns. The numerator of the expression for x is the determinant of the coefficient matrix of the system with the first column replaced by the constant terms. Similarly, the numerator of the expression for y is the determinant of the coefficient matrix of the system with the second column replaced by the constant terms.

In Subsection 5.4 we will prove that a 2×2 matrix is invertible if and only if its determinant is non-zero. For an invertible 2×2 matrix, there is a quick way to find the inverse using the determinant. You can verify the following strategy by checking that the expression given below for $\mathbf{A}^{-1}$ does indeed satisfy $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$.

Strategy C4

To find the inverse of a 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $\det \mathbf{A} = ad - bc \neq 0$:

- interchange the diagonal entries
- multiply the non-diagonal entries by -1
- divide by the determinant of $\mathbf{A}$, giving

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Exercise C36

Determine whether or not each of the following matrices is invertible, and find the inverse if it exists.

(a) $\begin{pmatrix} 4 & 2 \\ 5 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

There is also a geometric interpretation of the determinant: let (a, c) and (b, d) be two position vectors. Then the determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

gives the area of the parallelogram with adjacent sides given by these position vectors. For example, the parallelogram shown in Figure 16 with vertices $(0, 0)$, $(2, 1)$, $(1, 3)$ and $(3, 4)$ has area 5, since the base and height are both equal to $\sqrt{5}$. Now, since one of the vertices is at the origin, the position vectors $(2, 1)$ and $(1, 3)$ determine the parallelogram, and

$$\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = (2 \times 3) - (1 \times 1) = 5.$$

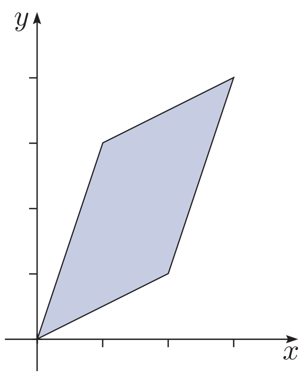


Figure 16 A parallelogram with area 5

Determinant of a 3×3 matrix

We now consider the following system of three linear equations in three unknowns:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \quad \text{or} \quad \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

Again we can find the solution, if one exists, using Gauss–Jordan elimination. It turns out that the solutions for x, y and z all have the same denominator:

$$a_1b_2c_3 - a_1c_2b_3 - b_1a_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3.$$

This is the *determinant* of the 3×3 coefficient matrix. Notice that each term in this expression for the denominator contains the letters a, b and c, and the subscripts 1, 2 and 3, in some order.

The definition we give for the determinant of a 3×3 matrix is expressed in terms of three 2×2 determinants. This is the easiest way to remember the definition.

Definition

The **determinant** of a 3×3 matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

is

$$\det \mathbf{A} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

Notice the minus sign before the second term on the right-hand side.

Worked Exercise C17

Evaluate the determinant of each of the following 3×3 matrices.

(a) $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & -1 \\ -2 & 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 0 & 1 \\ 0 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$

Solution

(a)
$$\begin{aligned} \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & -1 \\ -2 & 1 & 1 \end{vmatrix} &= 1 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} \\ &= 1\left((1 \times 1) - (-1 \times 1)\right) \\ &\quad - 2\left((3 \times 1) - (-1 \times (-2))\right) \\ &\quad + 1\left((3 \times 1) - (1 \times (-2))\right) \\ &= 5 \end{aligned}$$

(b)
$$\begin{aligned} \begin{vmatrix} 4 & 0 & 1 \\ 0 & -1 & 2 \\ 2 & 1 & 3 \end{vmatrix} &= 4 \begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} - 0 \begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} + 1 \begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} \\ &= 4\left((-1 \times 3) - (2 \times 1)\right) \\ &\quad - 0 + 1\left((0 \times 1) - (-1 \times 2)\right) \\ &= -18 \end{aligned}$$

Exercise C37

Evaluate the determinant of each of the following 3×3 matrices.

(a) $\begin{pmatrix} 3 & 2 & 1 \\ 4 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 10 & 0 \\ 3 & -1 & 2 \\ 5 & 9 & 2 \end{pmatrix}$

Determinants of larger matrices (4×4, and so on) are defined similarly in terms of smaller determinants in the next subsection. Note that determinants are defined only for square matrices. As with 2×2 matrices, determinants of larger matrices can be used to solve systems of linear equations.

5.2 Evaluating determinants

You have seen that although the determinant of a 2×2 matrix is simple to evaluate, the determinant of a 3×3 matrix is quite complicated. Determinants of larger matrices become increasingly more complicated as the size of the matrix increases. You will mainly be finding determinants of matrices of size 2×2 and 3×3. In this subsection we develop a strategy for evaluating determinants by expressing them eventually in terms of determinants of 2×2 matrices, as with the definition of the determinant of a 3×3 matrix above.

Cofactors

A **submatrix** is a matrix formed from another matrix with some of the rows and/or columns removed; submatrices are useful when evaluating determinants.

We can express the determinant of a 3×3 matrix $\mathbf{A} = (a_{ij})$ conveniently as

$$\det \mathbf{A} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$$

The elements A_{11}, A_{12} and A_{13} in this expression are called the *cofactors* of the elements a_{11}, a_{12} and a_{13}, respectively. We can see from the definition of the determinant that these cofactors are themselves determinants with a $+$ or $-$ sign attached. In fact, A_{1j} is $(-1)^{1+j}$ times the determinant of a submatrix of $\mathbf{A}$ formed by removing the top row and one column of $\mathbf{A}$ – namely the row and column containing the element a_{1j}.

Thus for A_{11} we have

$$(-1)^{1+1}\begin{vmatrix} a_{11} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{vmatrix} \quad \text{so} \quad A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},$$

for A_{12} we have

$$(-1)^{1+2}\begin{vmatrix} \cancel{a_{11}} & a_{12} & \cancel{a_{13}} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{vmatrix} \quad \text{so} \quad A_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

and for A_{13} we have

$$(-1)^{1+3}\begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & a_{13} \\ a_{21} & a_{22} & \cancel{a_{23}} \\ a_{31} & a_{32} & \cancel{a_{33}} \end{vmatrix} \quad \text{so} \quad A_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

In fact, there is a *cofactor* associated with each entry of any square matrix.

Definition

Let $\mathbf{A} = (a_{ij})$ be an $n \times n$ matrix. The **cofactor** A_{ij} associated with the entry a_{ij} is

$$A_{ij} = (-1)^{i+j} \det \mathbf{A}_{ij},$$

where $\mathbf{A}_{ij}$ is the $(n-1) \times (n-1)$ submatrix of $\mathbf{A}$ resulting when the ith row and jth column (the row and column containing the entry a_{ij}) are removed.

For example, for the cofactor A_{23} of the 4×4 matrix $\mathbf{A} = (a_{ij})$ we have

$$(-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} & \cancel{a_{13}} & a_{14} \\ \cancel{a_{21}} & \cancel{a_{22}} & a_{23} & \cancel{a_{24}} \\ a_{31} & a_{32} & \cancel{a_{33}} & a_{34} \\ a_{41} & a_{42} & \cancel{a_{43}} & a_{44} \end{vmatrix} \quad \text{so} \quad A_{23} = -\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}.$$

Exercise C38

Write down expressions for the cofactors A_{13} and A_{45} of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}.$$

(Do not attempt to evaluate these expressions!)

Determinant of an $n \times n$ matrix

You have seen that we can use cofactors to evaluate the determinant of a 3×3 matrix. Determinants of larger matrices can be evaluated in a similar way.

Definition

The **determinant** of an $n \times n$ matrix $\mathbf{A} = (a_{ij})$ is

$$\begin{aligned} \det \mathbf{A} &= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \\ &= a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}. \end{aligned}$$

Do not forget the minus sign that is a part of alternate cofactors!

The determinant of a matrix is a complicated string of terms. The definition above collects the terms into manageable expressions using the cofactors of the entries of the top row; when we write the determinant in this way, we say that we are *expanding along the top row*.

There are alternative expansions for the determinant of a square matrix that collect the terms in different ways – however, the resulting value for the determinant is always the same.

We are now in a position to evaluate the determinant of any square matrix using the following strategy.

Strategy C5

To evaluate the determinant of an $n \times n$ matrix:

1. expand along the top row to express the $n \times n$ determinant in terms of n determinants of size $(n-1) \times (n-1)$
2. expand along the top row of each of the resulting determinants
3. repeatedly apply step 2 until the only determinants in the expression are of size 2×2
4. evaluate the final expression.

Figure 17 gives a scheme for an $n \times n$ determinant.

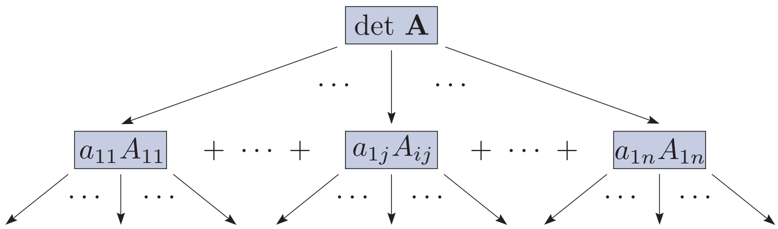


Figure 17 Scheme for an $n \times n$ determinant

Worked Exercise C18 illustrates Strategy C5, before you are asked to find the determinant of a 4×4 matrix in Exercise C39.

Worked Exercise C18

Evaluate the following determinant.

$$\begin{vmatrix} 2 & 0 & 3 & 5 \\ 0 & 4 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 \end{vmatrix}$$

Solution

We apply Strategy C5:

$$\begin{aligned}
\begin{vmatrix} 2 & 0 & 3 & 5 \\ 0 & 4 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 \end{vmatrix}
&= 2\begin{vmatrix} 4 & -1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix} - 0 + 3\begin{vmatrix} 0 & 4 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} - 5\begin{vmatrix} 0 & 4 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{vmatrix} \\
&= 2\left(4\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + 0\right) \\
&\quad + 3\left(0 - 4\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + 0\right) \\
&\quad - 5\left(0 - 4\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}\right) \\
&= 2(-4-2) + 3(-4) - 5(-4-2) \\
&= -12 - 12 + 30 \\
&= 6.
\end{aligned}$$

Exercise C39

Evaluate the following determinant.

$$\begin{vmatrix} 0 & 2 & 1 & -1 \\ -3 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 4 & 2 & 0 \end{vmatrix}$$

5.3 Properties of determinants

Suppose that $\mathbf{A}$ and $\mathbf{B}$ are two $n \times n$ matrices. Are there any relationships between $\det \mathbf{A}$, $\det \mathbf{B}$, $\det(\mathbf{A}+\mathbf{B})$ and $\det(\mathbf{AB})$?

Exercise C40

Let $\mathbf{A} = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ -2 & 5 \end{pmatrix}$.

Evaluate $\det \mathbf{A}$, $\det \mathbf{B}$, $\det(\mathbf{A}+\mathbf{B})$, $\det(\mathbf{AB})$ and $(\det \mathbf{A})(\det \mathbf{B})$.

Comment on your results.

You should have found in the solution to Exercise C40 that there does not appear to be a simple relationship for the addition of determinants; that is, we cannot easily express $\det(\mathbf{A}+\mathbf{B})$ in terms of $\det \mathbf{A}$ and $\det \mathbf{B}$. However, the identity

$$\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$$

does hold for all square matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size. The simplicity of this result is somewhat surprising, given the complexity of the definitions of matrix multiplication and the determinant.

We collect together, without proof, some results about determinants in the following theorem.

Theorem C14

Let $\mathbf{A}$ and $\mathbf{B}$ be two square matrices of the same size. Then the following hold:

(a) $\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$

(b) $\det \mathbf{I} = 1$

(c) $\det \mathbf{A}^T = \det \mathbf{A}$.

Elementary operations and determinants

Earlier, in Theorem C10, you saw that multiplication on the left by an *elementary matrix* has the same effect as applying the associated elementary row operation. Here, we use elementary matrices to prove some useful results about determinants.

Exercise C41

Evaluate the following determinants, where k is any real number.

(a) $\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} 1 & 0 \\ k & 1 \end{vmatrix}$

The results of Exercise C41 are particular cases of the following theorem. The proof is not hard, but it is not very illuminating, so is not given here.

Theorem C15

Let $\mathbf{E}$ be an elementary matrix, and let k be a non-zero real number.

(a) If $\mathbf{E}$ results from interchanging two rows of $\mathbf{I}$, then $\det \mathbf{E} = -1$.

(b) If $\mathbf{E}$ results from multiplying a row of $\mathbf{I}$ by k, then $\det \mathbf{E} = k$.

(c) If $\mathbf{E}$ results from adding k times one row of $\mathbf{I}$ to another row, then $\det \mathbf{E} = 1$.

Zeros in a matrix greatly simplify the calculation of the determinant. If an entire row of the matrix is zero, then all the terms vanish and the determinant is zero. Some other matrices with zero determinant are also easy to recognise.

Theorem C16

Let $\mathbf{A}$ be a square matrix. Then $\det \mathbf{A} = 0$ if any of the following hold:

(a) $\mathbf{A}$ has an entire row (or column) of zeros

(b) $\mathbf{A}$ has two equal rows (or columns)

(c) $\mathbf{A}$ has two proportional rows (or columns).

Proof We prove the statements for rows. The results for columns follow, as Theorem C14(c) states that taking the transpose does not alter the determinant of a matrix.

(a) We follow Strategy C5 and expand along the top row of $\mathbf{A}$, and continue by expanding along the top row of the resulting determinants until the only determinants in the expression are of size 2×2. The first term of the full expansion is therefore the product $a_{11}a_{22}a_{33} \cdots a_{nn}$, and each other term similarly comprises a product containing one entry from each row and each column.

Each term in the full expansion of the determinant of $\mathbf{A}$ is a product containing one entry from each row and each column of $\mathbf{A}$. If an entire row of $\mathbf{A}$ is zero, then each term of this expansion contains at least one zero, and so each term is zero. Hence the determinant of $\mathbf{A}$ is equal to zero.

(b) If the ith and jth rows of the matrix $\mathbf{A}$ are equal, then $\mathbf{A}$ remains the same if these rows are interchanged. Let $\mathbf{E}$ be the elementary matrix obtained by interchanging the ith and jth rows of $\mathbf{I}$. Then $\mathbf{EA} = \mathbf{A}$. Using Theorems C14 and C15, we have

$$\det \mathbf{A} = \det(\mathbf{EA}) = (\det \mathbf{E})(\det \mathbf{A}) = -1 \times \det \mathbf{A}.$$

This implies that $\det \mathbf{A} = 0$, as required.

(c) Two rows (or columns) of a matrix are *proportional* when one is a multiple of the other.

Suppose that the ith row of $\mathbf{A}$ is equal to k times the jth row. Let $\mathbf{E}$ be the elementary matrix obtained from $\mathbf{I}$ by multiplying the ith row by $1/k$. Then the ith and jth rows of the matrix $\mathbf{EA}$ are equal. The determinant of this matrix $\mathbf{EA}$ is zero, by (b) above. Using Theorem C14, we have

$$(\det \mathbf{E})(\det \mathbf{A}) = \det(\mathbf{EA}) = 0.$$

Now $\det \mathbf{E} = 1/k$, by Theorem C15. This implies that $\det \mathbf{A} = 0$, as required. ■

Exercise C42

Evaluate the determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 13 & 11 \\ -2 & 4 & -8 \end{pmatrix}.$$

Theorem C15(a) and Theorem C14(a) together mean that if $\mathbf{B}$ is a matrix obtained from a matrix $\mathbf{A}$ by interchanging a pair of rows, then $\det \mathbf{B} = -\det \mathbf{A}$. Therefore the evaluation of the determinant can be significantly simplified if a row of the matrix contains some zeros, as the following worked exercise illustrates.

Worked Exercise C19

Evaluate the determinant of the following matrix.

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 7 & 0 \\ 3 & 1 & 5 \end{pmatrix}$$

Solution

The second row of $\mathbf{A}$ has only one non-zero entry and so interchanging the top two rows will give only one non-zero term in the expansion.

We interchange the top two rows of $\mathbf{A}$, and apply Theorems C14 and C15, giving

$$\det \mathbf{A} = \begin{vmatrix} 1 & 4 & 2 \\ 0 & 7 & 0 \\ 3 & 1 & 5 \end{vmatrix} = (-1) \begin{vmatrix} 0 & 7 & 0 \\ 1 & 4 & 2 \\ 3 & 1 & 5 \end{vmatrix}.$$

We use Strategy C5 to evaluate this determinant:

$$\begin{aligned} \det \mathbf{A} &= (-1) \left(0 - 7 \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} + 0 \right) \\ &= 7\left((1 \times 5) - (2 \times 3)\right) \\ &= -7. \end{aligned}$$

Exercise C43

Evaluate the determinant of the following matrix.

$$\mathbf{A} = \begin{pmatrix} 10 & 3 & -4 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 6 & 0 & 0 \\ -1 & 2 & 1 & 0 \end{pmatrix}$$

5.4 Determinants and inverses of matrices

Earlier, in Subsection 5.1, we stated that the inverse of a 2×2 matrix $\mathbf{A}$ exists if and only if $\det \mathbf{A} \neq 0$. This extends to all square matrices.

Theorem C17

A square matrix $\mathbf{A}$ is invertible if and only if $\det \mathbf{A} \neq 0$.

Proof Let $\mathbf{A}$ be an $n \times n$ matrix.

We start by proving the *only if* statement.

First we show that if $\mathbf{A}$ is invertible, then $\det \mathbf{A} \neq 0$.

Suppose that $\mathbf{A}$ is invertible. Then since $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n$, it follows from Theorem C14 that

$$(\det \mathbf{A})(\det \mathbf{A}^{-1}) = \det(\mathbf{A}\mathbf{A}^{-1}) = \det \mathbf{I}_n = 1.$$

Therefore neither $\det \mathbf{A}$ nor $\det \mathbf{A}^{-1}$ is 0.

We now prove the *if* statement.

Next we show that if $\det \mathbf{A} \neq 0$, then $\mathbf{A}$ is invertible.

Now suppose that $\det \mathbf{A} \neq 0$. Let $\mathbf{E}_1, \dots, \mathbf{E}_k$ be elementary matrices such that $\mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{U}$ is matrix $\mathbf{A}$ in row-reduced form. Using Theorems C14 and C15 and the assumption that $\det \mathbf{A} \neq 0$, we have

$$\det \mathbf{U} = (\det \mathbf{E}_k) \cdots (\det \mathbf{E}_2)(\det \mathbf{E}_1)(\det \mathbf{A}) \neq 0.$$

Now this implies that $\mathbf{U}$ has no zero row, and therefore has a leading 1 in each of its n rows. Hence $\mathbf{U} = \mathbf{I}_n$, and so, by the Invertibility Theorem (Theorem C7), the matrix $\mathbf{A}$ is invertible, with

$$\mathbf{A}^{-1} = \mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1.$$

■

We saw in the proof of Theorem C17 above that if $\mathbf{A}$ is invertible, then $(\det \mathbf{A})(\det \mathbf{A}^{-1}) = 1$. This gives the following useful result.

$$\det \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}$$

Until now, if we wanted to show that an $n \times n$ matrix $\mathbf{A}$ is invertible, we had to produce an $n \times n$ matrix $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{I} = \mathbf{BA}.$$

The next theorem shows that if one of these conditions holds, then the other holds automatically. Thus if we want to show that an $n \times n$ matrix $\mathbf{A}$ is invertible, it is enough to produce an $n \times n$ matrix $\mathbf{B}$ satisfying *either* condition.

Theorem C18

Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices of the same size. Then $\mathbf{AB} = \mathbf{I}$ if and only if $\mathbf{BA} = \mathbf{I}$.

Proof We start by proving the *only if* statement.

First we show that if $\mathbf{AB} = \mathbf{I}$, then $\mathbf{BA} = \mathbf{I}$.

Suppose that $\mathbf{AB} = \mathbf{I}$. Then, by Theorem C14,

$$(\det \mathbf{A})(\det \mathbf{B}) = \det(\mathbf{AB}) = \det \mathbf{I} = 1.$$

This implies that

$$\det \mathbf{A} \neq 0 \quad \text{and} \quad \det \mathbf{B} \neq 0,$$

so, by Theorem C17, $\mathbf{A}$ and $\mathbf{B}$ are both invertible.

Now,

$$\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{I},$$

and we can write $\mathbf{I}$ as $\mathbf{AB}$, so

$$\mathbf{A}^{-1} = \mathbf{A}^{-1}(\mathbf{AB}) = (\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{IB} = \mathbf{B},$$

and therefore

$$\mathbf{BA} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I},$$

as required.

To prove the *if* statement we have to show that if $\mathbf{BA} = \mathbf{I}$, then $\mathbf{AB} = \mathbf{I}$. We can use exactly the same argument as above with $\mathbf{A}$ and $\mathbf{B}$ exchanged.

The same argument, with the roles of $\mathbf{A}$ and $\mathbf{B}$ interchanged, proves the converse. ■

We summarise the results on the invertibility of a matrix $\mathbf{A}$ as follows. This one theorem collects together Theorems C7, C9 and C17.

Theorem C19 Summary Theorem

Let $\mathbf{A}$ be an $n \times n$ matrix. Then the following statements are equivalent.

(a) $\mathbf{A}$ is invertible.

(b) $\det \mathbf{A} \neq 0$.

(c) The row-reduced form of $\mathbf{A}$ is $\mathbf{I}_n$.

(d) The system $\mathbf{Ax} = \mathbf{b}$ has precisely one solution for each $n \times 1$ matrix $\mathbf{b}$.

(e) The system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.

To conclude this section, we collect together some of the most important properties of matrices from this unit.

Summary of properties of matrices

Let $\mathbf{A}$ and $\mathbf{B}$ be two square matrices of the same size. Then

$$\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B}),$$

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T,$$

$$\det \mathbf{A}^T = \det \mathbf{A}.$$

If $\mathbf{A}$ and $\mathbf{B}$ are invertible, then

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1},$$

$$\det \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}.$$

Summary

In this unit you have seen that systems of linear equations can have no solution, a unique solution or infinitely many solutions, and you have used Gauss–Jordan elimination to solve such systems. You have seen that matrices can be used in two different ways to represent systems of linear equations: both as an augmented matrix and as a matrix equation in which the coefficient matrix is multiplied on the right by the matrix of unknowns to give the matrix of constant terms. You studied how properties of matrices relate to properties of the corresponding systems of linear equations. In particular, you saw that if the coefficient matrix of a system of linear equations is invertible, or equivalently, if the determinant of the coefficient matrix is non-zero, then the system has a unique solution. You also saw that the set of $m \times n$ matrices with real entries forms an abelian group under addition and that the set of $n \times n$ invertible matrices with real entries forms an abelian group under matrix multiplication.

You will encounter systems of linear equations throughout the linear algebra units, along with matrices and their properties. Matrices will also appear in the group theory units, in particular you will work with the group of invertible 2×2 matrices in Book E.

Learning outcomes

After working through this unit, you should be able to:

- understand the connection between the solutions of systems of linear equations in two and three unknowns, and the intersection of lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$
- explain the terms *solution set*, *consistent*, *inconsistent* and *homogeneous system of linear equations*
- use the method of *Gauss–Jordan elimination* to find the solutions of systems of linear equations
- describe the three types of *elementary operation* and *elementary row operation*
- recognise whether or not a given matrix is in *row-reduced form* and row-reduce a matrix
- write down the *augmented matrix* of a system of linear equations, recover a system of linear equations from its augmented matrix, and solve a system of linear equations by row-reducing its augmented matrix
- perform the matrix operations of addition, multiplication and transposition
- recognise the following types of matrix: *square*, *zero*, *diagonal*, *lower triangular*, *upper triangular*, *identity*, *symmetric*
- express a system of linear equations in *matrix form* and state the relationship between the invertibility of the coefficient matrix and the number of solutions of the system
- understand what is meant by an *invertible* matrix and determine whether or not a given matrix is invertible and, if it is, find its inverse
- understand that the set of $n \times n$ invertible matrices with real entries forms a group under matrix multiplication
- understand the connections between elementary row operations and elementary matrices
- understand the term *determinant* of a square matrix, evaluate the determinant of a 2×2 matrix and *expand along the top row* to calculate the determinant of larger matrices
- use determinants to check whether or not a matrix is invertible.

Solutions to exercises

Solution to Exercise C1

(a) This is a linear equation.

(b) This is not a linear equation. The third term involves the product of x_3 and x_4.

(c) This is a linear equation (although not all of the five unknowns appear in this equation).

(d) This is not a linear equation. For example, the second term, $a_2x_2^2$, involves a product of unknowns.

Solution to Exercise C2

A general homogeneous system of m linear equations in n unknowns is

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad\qquad \vdots \quad &\ \ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

We substitute the values $x_1 = 0$, $x_2 = 0$, $\ldots$, $x_n = 0$ into the equations of the system:

$$\begin{aligned} a_{11}0 + \cdots + a_{1n}0 &= 0 \\ a_{21}0 + \cdots + a_{2n}0 &= 0 \\ \vdots \qquad\qquad \vdots \quad &\ \ \vdots \\ a_{m1}0 + \cdots + a_{mn}0 &= 0. \end{aligned}$$

All the equations are satisfied, whatever the values of the coefficients a_{ij}. The solution set therefore contains the trivial solution

$$x_1 = 0,\ x_2 = 0,\ \ldots,\ x_n = 0.$$

Solution to Exercise C3

We label the equations and apply elementary operations to simplify the system.

$$\begin{array}{ll} \mathbf{r}_1 & x + y = 4 \\ \mathbf{r}_2 & 2x - y = 5 \end{array}$$

First we eliminate the unknown x from the second equation.

$$\begin{array}{ll} & x + y = 4 \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 & \quad - 3y = -3 \end{array}$$

We then simplify this equation.

$$\begin{array}{ll} & x + y = 4 \\ \mathbf{r}_2 \to -\frac{1}{3}\mathbf{r}_2 & \qquad y = 1 \end{array}$$

We use it to eliminate the unknown y from the first equation of the system.

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 & x = 3 \\ & y = 1 \end{array}$$

We conclude that there is a unique solution: $x = 3$, $y = 1$.

The method above eliminates the unknowns in order; you may have begun by performing the elementary operation $\mathbf{r}_1 \to \mathbf{r}_1 + \mathbf{r}_2$ to eliminate y from $\mathbf{r}_1$. This is also correct.

Solution to Exercise C4

The explanations in between the systems of three linear equations are not a necessary part of the solution: they are included for clarity.

We label the equations and apply elementary operations to simplify the system.

$$\begin{array}{ll} \mathbf{r}_1 & x + y - z = 8 \\ \mathbf{r}_2 & 2x - y + z = 1 \\ \mathbf{r}_3 & -x + 3y + 2z = -8 \end{array}$$

First use the $\mathbf{r}_1$ equation to eliminate the unknown x from the other equations.

$$\begin{array}{ll} & x + y - z = 8 \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 & \quad - 3y + 3z = -15 \\ \mathbf{r}_3 \to \mathbf{r}_3 + \mathbf{r}_1 & \qquad 4y + z = 0 \end{array}$$

Now simplify $\mathbf{r}_2$.

$$\begin{array}{ll} & x + y - z = 8 \\ \mathbf{r}_2 \to -\frac{1}{3}\mathbf{r}_2 & \qquad y - z = 5 \\ & \qquad 4y + z = 0 \end{array}$$

Then use $\mathbf{r}_2$ to eliminate the y-terms from $\mathbf{r}_1$ and $\mathbf{r}_3$.

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 & x \qquad\quad = 3 \\ & \quad y - z = 5 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 4\mathbf{r}_2 & \qquad\ 5z = -20 \end{array}$$

Now simplify $\mathbf{r}_3$.

$$\begin{array}{ll} & x \qquad = 3 \\ & y - z = 5 \\ \mathbf{r}_3 \to \frac{1}{5}\mathbf{r}_3 & z = -4 \end{array}$$

Then use $\mathbf{r}_3$ to eliminate the z-term from $\mathbf{r}_2$.

$$\begin{array}{ll} & x = 3 \\ \mathbf{r}_2 \to \mathbf{r}_2 + \mathbf{r}_3 & y = 1 \\ & z = -4 \end{array}$$

We conclude that there is a unique solution: $x = 3$, $y = 1$, $z = -4$.

Solution to Exercise C5

We label the equations and apply elementary operations to simplify the system.

$$\begin{array}{ll} \mathbf{r}_1 & x + 3y - z = 4 \\ \mathbf{r}_2 & -x + 2y - 4z = 6 \\ \mathbf{r}_3 & x + 2y \qquad = 2 \end{array}$$

$$\begin{array}{ll} & x + 3y - z = 4 \\ \mathbf{r}_2 \to \mathbf{r}_2 + \mathbf{r}_1 & 5y - 5z = 10 \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1 & -y + z = -2 \end{array}$$

$$\begin{array}{ll} & x + 3y - z = 4 \\ \mathbf{r}_2 \to \frac{1}{5}\mathbf{r}_2 & y - z = 2 \\ & -y + z = -2 \end{array}$$

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - 3\mathbf{r}_2 & x \quad + 2z = -2 \\ & y - z = 2 \\ \mathbf{r}_3 \to \mathbf{r}_3 + \mathbf{r}_2 & 0 = 0 \end{array}$$

(The $\mathbf{r}_3$ equation $(0 = 0)$ gives no constraints on x, y and z.)

There are insufficient constraints on the unknowns to determine them uniquely, so the system has an infinite solution set.

(As both remaining equations involve a z-term, set z equal to the real parameter k.)

We write the general solution as

$$x = -2 - 2k, \quad y = 2 + k, \quad z = k, \quad k \in \mathbb{R}.$$

You may have spotted that $\mathbf{r}_2$ $(y - z = 2)$ and $\mathbf{r}_3$ $(-y + z = -2)$ were multiples of each other, and concluded earlier that there are infinitely many solutions; however, the solutions are still needed.

Solution to Exercise C6

We label the equations, and apply elementary operations to simplify the system.

$$\begin{array}{ll} \mathbf{r}_1 & x + y + z = 6 \\ \mathbf{r}_2 & -x + y - 3z = -2 \\ \mathbf{r}_3 & 2x + y + 3z = 6 \end{array}$$

$$\begin{array}{ll} & x + y + z = 6 \\ \mathbf{r}_2 \to \mathbf{r}_2 + \mathbf{r}_1 & 2y - 2z = 4 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 2\mathbf{r}_1 & -y + z = -6 \end{array}$$

$$\begin{array}{ll} & x + y + z = 6 \\ \mathbf{r}_2 \to \frac{1}{2}\mathbf{r}_2 & y - z = 2 \\ & -y + z = -6 \end{array}$$

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 & x \quad + 2z = 4 \\ & y - z = 2 \\ \mathbf{r}_3 \to \mathbf{r}_3 + \mathbf{r}_2 & 0 = -4 \end{array}$$

The $\mathbf{r}_3$ equation is $0 = -4$, so we conclude that the solution set is empty: the system is inconsistent.

You may have spotted that the system is inconsistent at an earlier stage and therefore stopped then.

Solution to Exercise C7

Let the equation of the plane be

$$ax + by + cz = d,$$

where a, b, c and d are real, and a, b and c are not all zero.

Substituting the points into the equation gives a system of three linear equations in the unknowns a, b and c. We label the equations and apply elementary operations to simplify the system.

$$\begin{array}{ll} \mathbf{r}_1 & a \quad + 2c = d \\ \mathbf{r}_2 & 3b + 4c = d \\ \mathbf{r}_3 & a + b + 3c = d \end{array}$$

$$\begin{array}{ll} & a \quad + 2c = d \\ & 3b + 4c = d \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1 & b + c = 0 \end{array}$$

$$\begin{array}{ll} & a \quad + 2c = d \\ \mathbf{r}_2 \to \mathbf{r}_2 - 3\mathbf{r}_3 & c = d \\ & b + c = 0 \end{array}$$

$$\begin{array}{ll} & a \phantom{{}+b} + 2c = d \\ \mathbf{r}_2 \leftrightarrow \mathbf{r}_3 & \phantom{a+{}} b + c = 0 \\ & c = d \end{array}$$

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_3 & a = -d \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_3 & b = -d \\ & c = d \end{array}$$

We conclude that this system has a unique solution (in terms of d): $a = -d,\ b = -d,\ c = d$.

We substitute these expressions into the equation for the plane to get

$$-dx - dy + dz = d.$$

Dividing through by $-d$ yields a simpler equation for the plane:

$$x + y - z = -1.$$

Solution to Exercise C8

The two unknowns are *my sister's age* and *my brother's age*; let us denote these by s and b (in years), respectively.

The first statement of the problem now translates to the equation

$$s + b = 40,$$

and the second statement to

$$b = s + 12.$$

We write these two equations in the usual form and label them.

$$\begin{array}{lr} \mathbf{r}_1 & s + b = 40 \\ \mathbf{r}_2 & -s + b = 12 \end{array}$$

We apply elementary operations to simplify this system.

$$\begin{array}{ll} & s + b = 40 \\ \mathbf{r}_2 \to \mathbf{r}_2 + \mathbf{r}_1 & \phantom{s+{}} 2b = 52 \end{array}$$

$$\begin{array}{ll} & s + b = 40 \\ \mathbf{r}_2 \to \frac{1}{2}\mathbf{r}_2 & \phantom{s+{}} b = 26 \end{array}$$

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 & s = 14 \\ & b = 26 \end{array}$$

The system has a unique solution: $s = 14$, $b = 26$.

The answer to the problem is that my sister is 14 years old.

Solution to Exercise C9

(a) The augmented matrix of the system is

$$\left(\begin{array}{ccc|c} 4 & -2 & 0 & -7 \\ 0 & 1 & 3 & 0 \\ 0 & -3 & 1 & 3 \end{array}\right).$$

(b) The corresponding system is

$$\begin{aligned} 2x + 3y \phantom{{}-7z} + 7w &= 1 \\ y - 7z \phantom{{}+7w} &= -1 \\ x \phantom{{}+3y} + 3z - w &= 2. \end{aligned}$$

Solution to Exercise C10

(a) This matrix is not row-reduced as it does not have property 3.

(b) This matrix is row-reduced.

(c) This matrix is not row-reduced as it does not have property 4.

Solution to Exercise C11

(a) The augmented matrix corresponds to the system

$$\begin{aligned} x_1 \phantom{{}+x_2} &= \tfrac{1}{3} \\ x_2 &= \tfrac{2}{3}. \end{aligned}$$

The solution is $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$.

(b) The augmented matrix corresponds to the system

$$\begin{aligned} x_1 \phantom{{}+x_2} + 6x_3 &= 0 \\ x_2 + 7x_3 &= 0 \\ 0 &= 1. \end{aligned}$$

The third equation cannot be satisfied, so there are no solutions.

(c) The augmented matrix corresponds to the system

$$\begin{aligned} x_1 + 3x_2 \phantom{{}+x_3} + 2x_4 \phantom{{}+x_5} &= -7, \\ x_3 - 3x_4 \phantom{{}+x_5} &= 8, \\ x_5 &= 11, \end{aligned}$$

that is,

$$x_1 = -7 - 3x_2 - 2x_4$$

$$x_3 = 8 + 3x_4$$

$$x_5 = 11.$$

Setting $x_2 = k$ and $x_4 = l$ ($k, l \in \mathbb{R}$), we obtain the general solution

$$\begin{aligned} x_1 &= -7 - 3k - 2l, \\ x_2 &= k, \\ x_3 &= 8 + 3l, \\ x_4 &= l, \\ x_5 &= 11. \end{aligned}$$

(d) The augmented matrix corresponds to the system

$$\begin{aligned} x_1 \qquad\quad + \ x_4 &= 0 \\ x_2 \quad\ + 4x_4 &= 3 \\ x_3 \qquad &= 0 \end{aligned}$$

that is,

$$\begin{aligned} x_1 &= -x_4, \\ x_2 &= 3 - 4x_4, \\ x_3 &= 0. \end{aligned}$$

Setting $x_4 = k$ ($k \in \mathbb{R}$), we obtain the general solution

$$\begin{aligned} x_1 &= -k, \\ x_2 &= 3 - 4k, \\ x_3 &= 0, \\ x_4 &= k. \end{aligned}$$

Solution to Exercise C12

(a) Strategy C1 gives the following sequence of row operations.

$$\begin{matrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \end{matrix} \quad \begin{pmatrix} 1 & 5 & 1 & 4 & 5 & -1 \\ 1 & 5 & 3 & 12 & 11 & 3 \\ 3 & 15 & -1 & -4 & 3 & -6 \\ -2 & -10 & 1 & 2 & -7 & 6 \end{pmatrix} \begin{matrix} 15 \\ 35 \\ 10 \\ -10 \end{matrix}$$

$$\begin{matrix} \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 3\mathbf{r}_1 \\ \mathbf{r}_4 \to \mathbf{r}_4 + 2\mathbf{r}_1 \end{matrix} \quad \begin{pmatrix} 1 & 5 & 1 & 4 & 5 & -1 \\ 0 & 0 & 2 & 8 & 6 & 4 \\ 0 & 0 & -4 & -16 & -12 & -3 \\ 0 & 0 & 3 & 10 & 3 & 4 \end{pmatrix} \begin{matrix} 15 \\ 20 \\ -35 \\ 20 \end{matrix}$$

$$\begin{matrix} \\ \mathbf{r}_2 \to \frac{1}{2}\mathbf{r}_2 \\ \\ \\ \end{matrix} \quad \begin{pmatrix} 1 & 5 & 1 & 4 & 5 & -1 \\ 0 & 0 & 1 & 4 & 3 & 2 \\ 0 & 0 & -4 & -16 & -12 & -3 \\ 0 & 0 & 3 & 10 & 3 & 4 \end{pmatrix} \begin{matrix} 15 \\ 10 \\ -35 \\ 20 \end{matrix}$$

$$\begin{matrix} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 + 4\mathbf{r}_2 \\ \mathbf{r}_4 \to \mathbf{r}_4 - 3\mathbf{r}_2 \end{matrix} \quad \begin{pmatrix} 1 & 5 & 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & 4 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & -2 & -6 & -2 \end{pmatrix} \begin{matrix} 5 \\ 10 \\ 5 \\ -10 \end{matrix}$$

$$\begin{matrix} \\ \\ \mathbf{r}_3 \leftrightarrow \mathbf{r}_4 \\ \\ \end{matrix} \quad \begin{pmatrix} 1 & 5 & 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & 4 & 3 & 2 \\ 0 & 0 & 0 & -2 & -6 & -2 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix} \begin{matrix} 5 \\ 10 \\ -10 \\ 5 \end{matrix}$$

$$\begin{matrix} \\ \\ \mathbf{r}_3 \to -\frac{1}{2}\mathbf{r}_3 \\ \\ \end{matrix} \quad \begin{pmatrix} 1 & 5 & 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & 4 & 3 & 2 \\ 0 & 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix} \begin{matrix} 5 \\ 10 \\ 5 \\ 5 \end{matrix}$$

$$\begin{matrix} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 4\mathbf{r}_3 \\ \\ \\ \end{matrix} \quad \begin{pmatrix} 1 & 5 & 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & 0 & -9 & -2 \\ 0 & 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix} \begin{matrix} 5 \\ -10 \\ 5 \\ 5 \end{matrix}$$

$$\begin{matrix} \\ \\ \\ \mathbf{r}_4 \to \frac{1}{5}\mathbf{r}_4 \end{matrix} \quad \begin{pmatrix} 1 & 5 & 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & 0 & -9 & -2 \\ 0 & 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} 5 \\ -10 \\ 5 \\ 1 \end{matrix}$$

$$\begin{matrix} \mathbf{r}_1 \to \mathbf{r}_1 + 3\mathbf{r}_4 \\ \mathbf{r}_2 \to \mathbf{r}_2 + 2\mathbf{r}_4 \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_4 \\ \\ \end{matrix} \quad \begin{pmatrix} 1 & 5 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & -9 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} 8 \\ -8 \\ 4 \\ 1 \end{matrix}$$

This is the row-reduced form of the matrix.

(b) Strategy C1 gives the following sequence of row operations.

Your sequence may differ from this because in the first step shown below (which corresponds to step 2 of the strategy) row 1 and row 5 are interchanged, whereas row 1 could have been interchanged with another row. However, your final row-reduced matrix should be the same as this one, since (as you will see in Theorem C1) the row-reduced form of a matrix is unique.

$$\begin{matrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \\ \mathbf{r}_5 \end{matrix} \quad \begin{pmatrix} 0 & -8 & 8 & -14 \\ -1 & 0 & -4 & -6 \\ -1 & 8 & -12 & 8 \\ 2 & 8 & 0 & 24 \\ 1 & 4 & 0 & 14 \end{pmatrix} \begin{matrix} -14 \\ -11 \\ 3 \\ 34 \\ 19 \end{matrix}$$

$$\begin{matrix} \mathbf{r}_1 \leftrightarrow \mathbf{r}_5 \\ \\ \\ \\ \\ \end{matrix} \quad \begin{pmatrix} 1 & 4 & 0 & 14 \\ -1 & 0 & -4 & -6 \\ -1 & 8 & -12 & 8 \\ 2 & 8 & 0 & 24 \\ 0 & -8 & 8 & -14 \end{pmatrix} \begin{matrix} 19 \\ -11 \\ 3 \\ 34 \\ -14 \end{matrix}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 + \mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 + \mathbf{r}_1 \\ \mathbf{r}_4 \to \mathbf{r}_4 - 2\mathbf{r}_1 \\ \\ \end{array} \begin{pmatrix} 1 & 4 & 0 & 14 \\ 0 & 4 & -4 & 8 \\ 0 & 12 & -12 & 22 \\ 0 & 0 & 0 & -4 \\ 0 & -8 & 8 & -14 \end{pmatrix} \begin{array}{r} 19 \\ 8 \\ 22 \\ -4 \\ -14 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \frac{1}{4}\mathbf{r}_2 \\ \\ \\ \\ \end{array} \begin{pmatrix} 1 & 4 & 0 & 14 \\ 0 & 1 & -1 & 2 \\ 0 & 12 & -12 & 22 \\ 0 & 0 & 0 & -4 \\ 0 & -8 & 8 & -14 \end{pmatrix} \begin{array}{r} 19 \\ 2 \\ 22 \\ -4 \\ -14 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 4\mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 - 12\mathbf{r}_2 \\ \\ \mathbf{r}_5 \to \mathbf{r}_5 + 8\mathbf{r}_2 \end{array} \begin{pmatrix} 1 & 0 & 4 & 6 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{array}{r} 11 \\ 2 \\ -2 \\ -4 \\ 2 \end{array}$$

$$\begin{array}{l} \\ \\ \mathbf{r}_3 \to -\frac{1}{2}\mathbf{r}_3 \\ \\ \\ \end{array} \begin{pmatrix} 1 & 0 & 4 & 6 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{array}{r} 11 \\ 2 \\ 1 \\ -4 \\ 2 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 6\mathbf{r}_3 \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_3 \\ \\ \mathbf{r}_4 \to \mathbf{r}_4 + 4\mathbf{r}_3 \\ \mathbf{r}_5 \to \mathbf{r}_5 - 2\mathbf{r}_3 \end{array} \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{r} 5 \\ 0 \\ 1 \\ 0 \\ 0 \end{array}$$

This is the row-reduced form of the matrix.

Solution to Exercise C13

We have

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_1 \\ \\ \end{array} \begin{pmatrix} 1 & 3 & 1 & 2 \\ -1 & 1 & 4 & 5 \\ 0 & 3 & 4 & 9 \end{pmatrix} \begin{array}{r} 7 \\ 9 \\ 16 \end{array}$$

The row operation has created a 1 in the correct position in the current row, but it is not a leading 1 because it has changed the 0 at the beginning of the row to -1. Performing this row operation has destroyed the progress made so far on the matrix: the first column no longer contains a leading 1 with only zeros above and below.

Solution to Exercise C14

We follow Strategy C2 and row-reduce the augmented matrix.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \left(\begin{array}{ccccc|c} 1 & -4 & -4 & 3 & 6 & 2 \\ 2 & -5 & -6 & 6 & 9 & 3 \\ 2 & 4 & 0 & 9 & 2 & 0 \end{array}\right) \begin{array}{r} 4 \\ 9 \\ 17 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 2\mathbf{r}_1 \end{array} \left(\begin{array}{ccccc|c} 1 & -4 & -4 & 3 & 6 & 2 \\ 0 & 3 & 2 & 0 & -3 & -1 \\ 0 & 12 & 8 & 3 & -10 & -4 \end{array}\right) \begin{array}{r} 4 \\ 1 \\ 9 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \frac{1}{3}\mathbf{r}_2 \\ \\ \end{array} \left(\begin{array}{ccccc|c} 1 & -4 & -4 & 3 & 6 & 2 \\ 0 & 1 & \frac{2}{3} & 0 & -1 & -\frac{1}{3} \\ 0 & 12 & 8 & 3 & -10 & -4 \end{array}\right) \begin{array}{r} 4 \\ \frac{1}{3} \\ 9 \end{array}$$

(Note that here we cannot find a row operation that could be performed instead of $\mathbf{r}_2 \to \frac{1}{3}\mathbf{r}_2$ to create a leading 1 while avoiding fractions.)

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 + 4\mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 - 12\mathbf{r}_2 \end{array} \left(\begin{array}{ccccc|c} 1 & 0 & -\frac{4}{3} & 3 & 2 & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & 0 & -1 & -\frac{1}{3} \\ 0 & 0 & 0 & 3 & 2 & 0 \end{array}\right) \begin{array}{r} \frac{16}{3} \\ \frac{1}{3} \\ 5 \end{array}$$

$$\begin{array}{l} \\ \\ \mathbf{r}_3 \to \frac{1}{3}\mathbf{r}_3 \end{array} \left(\begin{array}{ccccc|c} 1 & 0 & -\frac{4}{3} & 3 & 2 & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & 0 & -1 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{2}{3} & 0 \end{array}\right) \begin{array}{r} \frac{16}{3} \\ \frac{1}{3} \\ \frac{5}{3} \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 3\mathbf{r}_3 \\ \\ \\ \end{array} \left(\begin{array}{ccccc|c} 1 & 0 & -\frac{4}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & 0 & -1 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{2}{3} & 0 \end{array}\right) \begin{array}{r} \frac{1}{3} \\ \frac{1}{3} \\ \frac{5}{3} \end{array}$$

This matrix is in row-reduced form.
The corresponding system is

$$\begin{aligned} x_1 \quad - \tfrac{4}{3}x_3 \qquad\qquad &= \tfrac{2}{3} \\ x_2 + \tfrac{2}{3}x_3 \quad - \quad x_5 &= -\tfrac{1}{3} \\ x_4 + \tfrac{2}{3}x_5 &= 0, \end{aligned}$$

that is,

$$x_1 = \tfrac{2}{3} + \tfrac{4}{3}x_3$$

$$x_2 = -\tfrac{1}{3} - \tfrac{2}{3}x_3 + x_5$$

$$x_4 = -\tfrac{2}{3}x_5.$$

Setting $x_3 = k$ and $x_5 = l$ $(k, l \in \mathbb{R})$, we obtain the general solution

$$\begin{aligned} x_1 &= \tfrac{2}{3} + \tfrac{4}{3}k, \\ x_2 &= -\tfrac{1}{3} - \tfrac{2}{3}k + l, \\ x_3 &= k, \\ x_4 &= -\tfrac{2}{3}l, \\ x_5 &= l \quad (k, l \in \mathbb{R}). \end{aligned}$$

Solution to Exercise C15

(a) $\begin{pmatrix} 1 & -3 \\ -2 & 54 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 2 & 55 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -3 \\ -2 & 54 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 2 & 55 \end{pmatrix}$

(c) This sum is undefined since the matrices are of different sizes.

(d)
$$\begin{pmatrix} 0 & 6 & -2 \\ 1 & 8 & 2 \\ 0 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 9 \\ 1 & 0 & 4 \\ 3 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 8 & 7 \\ 2 & 8 & 6 \\ 3 & -1 & 5 \end{pmatrix}$$

Solution to Exercise C16

(a) This difference is undefined since the matrices are of different sizes.

(b)
$$\begin{pmatrix} 5 & 8 & 12 \\ 7 & 2 & -1 \end{pmatrix} - \begin{pmatrix} 3 & 10 & 2 \\ 4 & 9 & 21 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 10 \\ 3 & -7 & -22 \end{pmatrix}$$

Solution to Exercise C17

(a) $4\mathbf{A} = 4\begin{pmatrix} 5 & -3 \\ 2 & 3 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 20 & -12 \\ 8 & 12 \\ -4 & 0 \end{pmatrix}$

(b) $4\mathbf{B} = 4\begin{pmatrix} 2 & 1 \\ -2 & -7 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 8 & 4 \\ -8 & -28 \\ 12 & 20 \end{pmatrix}$

(c)
$$4\mathbf{A} + 4\mathbf{B} = \begin{pmatrix} 20 & -12 \\ 8 & 12 \\ -4 & 0 \end{pmatrix} + \begin{pmatrix} 8 & 4 \\ -8 & -28 \\ 12 & 20 \end{pmatrix} = \begin{pmatrix} 28 & -8 \\ 0 & -16 \\ 8 & 20 \end{pmatrix}$$

(d)
$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 5 & -3 \\ 2 & 3 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ -2 & -7 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 7 & -2 \\ 0 & -4 \\ 2 & 5 \end{pmatrix},$$

thus

$$4(\mathbf{A} + \mathbf{B}) = 4\begin{pmatrix} 7 & -2 \\ 0 & -4 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 28 & -8 \\ 0 & -16 \\ 8 & 20 \end{pmatrix}$$

(Note that $4(\mathbf{A} + \mathbf{B}) = 4\mathbf{A} + 4\mathbf{B}$.)

Solution to Exercise C18

(a) We add corresponding entries of the three matrices $\mathbf{A} = (a_{ij})$, $\mathbf{B} = (b_{ij})$ and $\mathbf{C} = (c_{ij})$. The (i, j)-entry of the matrix $\mathbf{A} + (\mathbf{B} + \mathbf{C})$ is $a_{ij} + (b_{ij} + c_{ij})$, and that of $(\mathbf{A} + \mathbf{B}) + \mathbf{C}$ is $(a_{ij} + b_{ij}) + c_{ij}$. Now, a_{ij}, b_{ij} and c_{ij} are real numbers, so $a_{ij} + (b_{ij} + c_{ij}) = (a_{ij} + b_{ij}) + c_{ij}$. Therefore

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}.$$

(b) We add corresponding entries of the two matrices. The (i, j)-entry of the matrix $\mathbf{A} + \mathbf{0}$ is $a_{ij} + 0 = a_{ij}$. Therefore $\mathbf{A} + \mathbf{0} = \mathbf{A}$.

Matrix addition is commutative (property A5), so $\mathbf{0} + \mathbf{A} = \mathbf{A}$ also.

(c) Let $\mathbf{A} = (a_{ij})$, so $-\mathbf{A} = (-a_{ij})$. We add corresponding entries: the (i, j)-entry of the matrix $\mathbf{A} + (-\mathbf{A})$ is $a_{ij} + (-a_{ij}) = 0$. Thus the matrix $\mathbf{A} + (-\mathbf{A})$ is the zero matrix $\mathbf{0}$.

Matrix addition is commutative (property A5), so $-\mathbf{A} + \mathbf{A} = \mathbf{A} + (-\mathbf{A})$. Thus $-\mathbf{A} + \mathbf{A}$ is also the zero matrix $\mathbf{0}$.

Solution to Exercise C19

Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$. Then the (i, j)-entry of $k(\mathbf{A} + \mathbf{B})$ is $k(a_{ij} + b_{ij})$.

Now, $k\mathbf{A} = (ka_{ij})$ and $k\mathbf{B} = (kb_{ij})$, so the (i, j)-entry of $k\mathbf{A} + k\mathbf{B}$ is $ka_{ij} + kb_{ij} = k(a_{ij} + b_{ij})$ since $a_{i,j}$, $b_{i,j}$ and k are real numbers.

The (i, j)-entries of $k(\mathbf{A} + \mathbf{B})$ and $k\mathbf{A} + k\mathbf{B}$ are equal. Thus

$$k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}.$$

Solution to Exercise C20

(a) The product of a 3×2 matrix with a 2×1 matrix is a 3×1 matrix:

$$\begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 7 \end{pmatrix}.$$

(b) The product of a 1×2 matrix with a 2×2 matrix is a 1×2 matrix:

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 14 \end{pmatrix}.$$

(c) This product is not defined, since the first matrix has 1 column and the second has 2 rows.

(d) The product of a 2×1 matrix with a 1×3 matrix is a 2×3 matrix:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & 0 & -4 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -4 \\ 6 & 0 & -8 \end{pmatrix}.$$

(e) The product of a 2×3 matrix with a 3×3 matrix is a 2×3 matrix:

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 1 \\ 1 & 3 & 0 \\ 4 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 5 & 1 \\ 9 & 16 & -1 \end{pmatrix}.$$

Solution to Exercise C21

(a) We first prove the *if* statement.

Suppose $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size, then the product $\mathbf{AB}$ can be formed because $\mathbf{A}$ has the same number of columns as $\mathbf{B}$ has rows. Likewise, the product $\mathbf{BA}$ can be formed. Both the products $\mathbf{AB}$ and $\mathbf{BA}$ will be square matrices the same size as $\mathbf{A}$ and $\mathbf{B}$.

We now prove the *only if* statement.

Suppose the products $\mathbf{AB}$ and $\mathbf{BA}$ are the same size, and suppose $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is a $p \times r$ matrix.

Since the product $\mathbf{AB}$ is defined, then we must have $n = p$, and therefore the product $\mathbf{AB}$ is an $m \times r$ matrix.

Since the product $\mathbf{BA}$ is defined, then we must have $r = m$, and therefore the product $\mathbf{BA}$ is a $p \times n$ matrix.

Since the products $\mathbf{AB}$ and $\mathbf{BA}$ are the same size, $m = p$ and $r = n$, but this combined with $n = p$ and $r = m$, implies that $p = r = m = n$. Therefore, both $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size.

(b) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$\mathbf{AB} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $\mathbf{BA} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so $\mathbf{AB} \neq \mathbf{BA}$ in this case. It follows that matrix multiplication is not commutative even for square matrices of the same size.

(There are infinitely many possible examples here; however, the trick when looking for a counterexample is to do as little work as possible: setting several of the entries to zero makes the multiplication easier!)

Solution to Exercise C22

The product of a 2×2 matrix with a 2×2 matrix is a 2×2 matrix.

(a) $\mathbf{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 28 \end{pmatrix}$

(b) $\mathbf{BA} = \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 28 \end{pmatrix}$

Note that $\mathbf{AB}$ and $\mathbf{BA}$ are equal in this case.

(c) Matrix multiplication is associative, so

$$\begin{aligned}
\mathbf{ABC} &= (\mathbf{AB})\mathbf{C} \\
&= \left(\begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix} \right) \begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix} \\
&= \begin{pmatrix} -3 & 0 \\ 0 & 28 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix} \\
&= \begin{pmatrix} -6 & 0 \\ 0 & 336 \end{pmatrix}.
\end{aligned}$$

(If you worked out $= \mathbf{A}(\mathbf{BC})$ then you would have got the same final answer here.)

Solution to Exercise C23

(a) The matrix $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ is upper triangular.

(b) The matrix $\begin{pmatrix} 9 & 0 \\ 0 & 0 \end{pmatrix}$ is diagonal (so it is also both upper and lower triangular).

(c) The matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ is *not* diagonal, upper triangular or lower triangular.

(d) The matrix $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is lower triangular.

Solution to Exercise C24

The (i, j)-entry of the product $\mathbf{I}_m\mathbf{A}$ is obtained by multiplying each entry in the ith row of $\mathbf{I}_m$ by the corresponding entry in the jth column of $\mathbf{A}$ and adding the results. Now, the ith row of $\mathbf{I}_m$ has a 1 in the ith position and zeros elsewhere. Therefore the only non-zero term in this sum is the ith entry of the jth column of $\mathbf{A}$, that is, the (i, j)-entry of $\mathbf{A}$. Thus $\mathbf{I}_m\mathbf{A} = \mathbf{A}$.

The (i, j)-entry of the product $\mathbf{A}\mathbf{I}_n$ is obtained by multiplying each entry in the ith row of $\mathbf{A}$ by the corresponding entry in the jth column of $\mathbf{I}_n$ and adding the results. Now, the jth column of $\mathbf{I}_n$ has a 1 in the jth position and zeros elsewhere. Therefore the only non-zero term in this sum is the jth entry of the ith row of $\mathbf{A}$, that is, the (i, j)-entry of $\mathbf{A}$. Thus $\mathbf{A}\mathbf{I}_n = \mathbf{A}$.

Solution to Exercise C25

(a) The transpose of a 3×2 matrix is a 2×3 matrix:

$$\begin{pmatrix} 1 & 4 \\ 0 & 2 \\ -6 & 10 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & -6 \\ 4 & 2 & 10 \end{pmatrix}.$$

(b) The transpose of a 3×3 matrix is a 3×3 matrix:

$$\begin{pmatrix} 2 & 1 & 2 \\ 0 & 3 & -5 \\ 4 & 7 & 0 \end{pmatrix}^T = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 3 & 7 \\ 2 & -5 & 0 \end{pmatrix}.$$

(c) The transpose of a 1×3 matrix is a 3×1 matrix:

$$\begin{pmatrix} 10 & 4 & 6 \end{pmatrix}^T = \begin{pmatrix} 10 \\ 4 \\ 6 \end{pmatrix}.$$

(d) The transpose of a 2×2 matrix is a 2×2 matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Solution to Exercise C26

(a) Here,

$$\mathbf{A}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix},$$

$$\mathbf{B}^T = \begin{pmatrix} 7 & 9 & 11 \\ 8 & 10 & 12 \end{pmatrix}$$

and

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 8 & 10 \\ 12 & 14 \\ 16 & 18 \end{pmatrix}.$$

So

$$(\mathbf{A} + \mathbf{B})^T = \begin{pmatrix} 8 & 12 & 16 \\ 10 & 14 & 18 \end{pmatrix}$$

and

$$\begin{aligned} \mathbf{A}^T + \mathbf{B}^T &= \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} + \begin{pmatrix} 7 & 9 & 11 \\ 8 & 10 & 12 \end{pmatrix} \\ &= \begin{pmatrix} 8 & 12 & 16 \\ 10 & 14 & 18 \end{pmatrix} \\ &= (\mathbf{A} + \mathbf{B})^T. \end{aligned}$$

(b) Here,

$$\mathbf{C}^T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$\mathbf{AC} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 7 & 4 \\ 11 & 6 \end{pmatrix}.$$

So

$$(\mathbf{AC})^T = \begin{pmatrix} 3 & 7 & 11 \\ 2 & 4 & 6 \end{pmatrix}.$$

The product $\mathbf{A}^T\mathbf{C}^T$ cannot be formed, since $\mathbf{A}^T$ is a 2×3 matrix and $\mathbf{C}^T$ is a 2×2 matrix.
The product $\mathbf{C}^T\mathbf{A}^T$ does, however, exist:

$$\begin{aligned} \mathbf{C}^T\mathbf{A}^T &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 7 & 11 \\ 2 & 4 & 6 \end{pmatrix} \\ &= (\mathbf{AC})^T. \end{aligned}$$

Solution to Exercise C27

Suppose, for a contradiction, that there exists a matrix $\mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\mathbf{AB} = \mathbf{I}$, that is,

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Multiplying the matrices on the left-hand side gives:

$$\begin{pmatrix} a-c & b-d \\ -a+c & -b+d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Looking at the entries in the first column, we must have $a-c=1$ and $-a+c=0$, that is, $a-c=1$ and $a-c=0$. This contradiction shows that there exists no such matrix $\mathbf{B}$. (The same conclusion arises from looking at the entries in the second column.)

Solution to Exercise C28

The equation $\mathbf{II} = \mathbf{I}$ shows that $\mathbf{I}$ is invertible, with inverse $\mathbf{I}$.

Solution to Exercise C29

To prove that $\mathbf{AB}$ is invertible, with inverse $\mathbf{B}^{-1}\mathbf{A}^{-1}$, we have to show that

$$(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{I} = (\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}).$$

By the associative property (M2),

$$\begin{aligned}(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) &= \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1} \\ &= \mathbf{AIA}^{-1} \\ &= \mathbf{AA}^{-1} = \mathbf{I},\end{aligned}$$

and, similarly,

$$\begin{aligned}(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) &= \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} \\ &= \mathbf{B}^{-1}\mathbf{IB} \\ &= \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}.\end{aligned}$$

Therefore $\mathbf{AB}$ is invertible, with inverse $\mathbf{B}^{-1}\mathbf{A}^{-1}$.

Solution to Exercise C30

(a) We row-reduce $(\mathbf{A} \mid \mathbf{I})$.

$$\begin{array}{ll} \mathbf{r}_1 \\ \mathbf{r}_2 \end{array} \quad \left(\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array}\right) \begin{array}{c} 7 \\ 6 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \frac{1}{2}\mathbf{r}_1 \\ \end{array} \quad \left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 4 & 1 & 0 & 1 \end{array}\right) \begin{array}{c} \frac{7}{2} \\ 6 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 4\mathbf{r}_1 \end{array} \quad \left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & -7 & -2 & 1 \end{array}\right) \begin{array}{c} \frac{7}{2} \\ -8 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to -\frac{1}{7}\mathbf{r}_2 \end{array} \quad \left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{2}{7} & -\frac{1}{7} \end{array}\right) \begin{array}{c} \frac{7}{2} \\ \frac{8}{7} \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_2 \\ \end{array} \quad \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{14} & \frac{2}{7} \\ 0 & 1 & \frac{2}{7} & -\frac{1}{7} \end{array}\right) \begin{array}{c} \frac{17}{14} \\ \frac{8}{7} \end{array}$$

The left half has been reduced to $\mathbf{I}$, so $\mathbf{A}$ is invertible; its inverse is

$$\mathbf{A}^{-1} = \begin{pmatrix} -\frac{1}{14} & \frac{2}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{pmatrix}.$$

(b) We row-reduce $(\mathbf{B} \mid \mathbf{I})$.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 1 & -4 & 1 & 0 & 0 \\ 2 & 1 & -6 & 0 & 1 & 0 \\ -3 & -1 & 9 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} -1 \\ -2 \\ 6 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 + 3\mathbf{r}_1 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 1 & -4 & 1 & 0 & 0 \\ 0 & -1 & 2 & -2 & 1 & 0 \\ 0 & 2 & -3 & 3 & 0 & 1 \end{array}\right) \begin{array}{c} -1 \\ 0 \\ 3 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to -\mathbf{r}_2 \\ \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & -1 & 0 \\ 0 & 2 & -3 & 3 & 0 & 1 \end{array}\right) \begin{array}{c} -1 \\ 0 \\ 3 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 - 2\mathbf{r}_2 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right) \begin{array}{c} -1 \\ 0 \\ 3 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 + 2\mathbf{r}_3 \\ \mathbf{r}_2 \to \mathbf{r}_2 + 2\mathbf{r}_3 \\ \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 5 & 2 \\ 0 & 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right) \begin{array}{c} 5 \\ 6 \\ 3 \end{array}$$

The left half has been reduced to $\mathbf{I}$, so $\mathbf{B}$ is invertible; its inverse is

$$\mathbf{B}^{-1} = \begin{pmatrix} -3 & 5 & 2 \\ 0 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix}.$$

(c) We row-reduce $(\mathbf{C} \mid \mathbf{I})$.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \quad \left(\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 5 & 10 & 5 & 0 & 0 & 1 \end{array}\right) \begin{array}{r} 13 \\ 8 \\ 21 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \frac{1}{2}\mathbf{r}_1 \\ \\ \\ \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & \frac{1}{2} & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 5 & 10 & 5 & 0 & 0 & 1 \end{array}\right) \begin{array}{r} \frac{13}{2} \\ 8 \\ 21 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 5\mathbf{r}_1 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & -10 & -\frac{5}{2} & 0 & 1 \end{array}\right) \begin{array}{r} \frac{13}{2} \\ \frac{3}{2} \\ -\frac{23}{2} \end{array}$$

The usual strategy for row-reduction has created a leading 1 in the second row that does not lie on the main diagonal of the left half. Hence the left half cannot reduce to $\mathbf{I}$, and therefore $\mathbf{C}$ is not invertible.

Solution to Exercise C31

The matrix form of the system is

$$\begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -4 \\ 3 & 2 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

Multiplying this equation on the left by the inverse of the coefficient matrix gives the solution

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 & -3 & -2 \\ -1 & 2 & 1 \\ -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -\frac{1}{2} \end{pmatrix};$$

that is, $x = 0$, $y = 2$, $z = -\frac{1}{2}$.

Solution to Exercise C32

(a) $$\underset{\substack{\text{elementary}\\ \text{matrix}\\ \text{associated with}\\ \mathbf{r}_1 \to 5\mathbf{r}_1}}{\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}} \underset{\mathbf{A}}{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}} = \underset{\substack{\text{matrix}\\ \text{obtained when}\\ \mathbf{r}_1 \to 5\mathbf{r}_1\\ \text{is performed on } \mathbf{A}}}{\begin{pmatrix} 5 & 10 & 15 \\ 3 & 2 & 1 \end{pmatrix}}$$

(b) $$\underset{\substack{\text{elementary}\\ \text{matrix}\\ \text{associated with}\\ \mathbf{r}_2 \to \mathbf{r}_2 + 3\mathbf{r}_4}}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \underset{\mathbf{B}}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}} = \underset{\substack{\text{matrix}\\ \text{obtained when}\\ \mathbf{r}_2 \to \mathbf{r}_2 + 3\mathbf{r}_4\\ \text{is performed on } \mathbf{B}}}{\begin{pmatrix} 1 & 2 \\ 24 & 28 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}}$$

Solution to Exercise C33

(a) The inverse elementary row operation of $\mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_2$ is $\mathbf{r}_1 \to \mathbf{r}_1 + 2\mathbf{r}_2$.

The working below shows the sequence of two row operations performed on $\mathbf{A}$.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \end{array} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_2 \\ \\ \end{array} \quad \begin{pmatrix} -7 & -8 & -9 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 + 2\mathbf{r}_2 \\ \\ \end{array} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

(b) The inverse elementary row operation of $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$ is $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$.

The working below shows the sequence of two row operations performed on $\mathbf{A}$.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \end{array} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\begin{array}{l} \mathbf{r}_1 \leftrightarrow \mathbf{r}_2 \\ \\ \end{array} \quad \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\begin{array}{l} \mathbf{r}_1 \leftrightarrow \mathbf{r}_2 \\ \\ \end{array} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

(c) The inverse elementary row operation of $\mathbf{r}_2 \to -3\mathbf{r}_2$ is $\mathbf{r}_2 \to -\frac{1}{3}\mathbf{r}_2$.

The working below shows the sequence of two row operations performed on $\mathbf{A}$.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \end{array} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to -3\mathbf{r}_2 \end{array} \quad \begin{pmatrix} 1 & 2 & 3 \\ -12 & -15 & -18 \end{pmatrix}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to -\frac{1}{3}\mathbf{r}_2 \end{array} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Solution to Exercise C34

Matrix $\mathbf{A}$ has associated elementary row operation $\mathbf{r}_1 \to 2\mathbf{r}_1$, which has inverse $\mathbf{r}_1 \to \frac{1}{2}\mathbf{r}_1$. The inverse of $\mathbf{A}$ is the elementary matrix associated with this inverse row operation, which is

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution to Exercise C35

(a) $\begin{vmatrix} 5 & 1 \\ 4 & 2 \end{vmatrix} = (5 \times 2) - (1 \times 4) = 6$

(b) $\begin{vmatrix} 10 & -4 \\ -5 & 2 \end{vmatrix} = (10 \times 2) - (-4 \times (-5)) = 0$

(c) $\begin{vmatrix} 7 & 3 \\ 17 & 2 \end{vmatrix} = (7 \times 2) - (3 \times 17) = -37$

Solution to Exercise C36

(a) First we evaluate the determinant of the matrix:

$$\begin{vmatrix} 4 & 2 \\ 5 & 6 \end{vmatrix} = (4 \times 6) - (2 \times 5) = 14.$$

This determinant is non-zero, so the matrix is invertible. We use the formula to find the inverse:

$$\begin{aligned} \begin{pmatrix} 4 & 2 \\ 5 & 6 \end{pmatrix}^{-1} &= \frac{1}{14}\begin{pmatrix} 6 & -2 \\ -5 & 4 \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{7} & -\frac{1}{7} \\ -\frac{5}{14} & \frac{2}{7} \end{pmatrix}. \end{aligned}$$

(b) First we evaluate the determinant of the matrix:

$$\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = (1 \times 1) - (1 \times (-1)) = 2.$$

This determinant is non-zero, so the matrix is invertible. We use the formula to find the inverse:

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

(c) First we evaluate the determinant of the matrix:

$$\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = (1 \times 1) - (-1 \times (-1)) = 0.$$

This determinant is 0, so the matrix is not invertible.

Solution to Exercise C37

(a) We have

$$\begin{aligned} &\begin{vmatrix} 3 & 2 & 1 \\ 4 & 0 & -1 \\ 0 & -1 & 1 \end{vmatrix} \\ &\quad = 3\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} 4 & -1 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & 0 \\ 0 & -1 \end{vmatrix} \\ &\quad = 3((0 \times 1) - (-1 \times (-1))) \\ &\qquad - 2((4 \times 1) - (-1 \times 0)) \\ &\qquad +((4 \times (-1)) - (0 \times 0)) \\ &\quad = -15 \end{aligned}$$

(b) We have

$$\begin{aligned} \begin{vmatrix} 2 & 10 & 0 \\ 3 & -1 & 2 \\ 5 & 9 & 2 \end{vmatrix} &= 2\begin{vmatrix} -1 & 2 \\ 9 & 2 \end{vmatrix} - 10\begin{vmatrix} 3 & 2 \\ 5 & 2 \end{vmatrix} + 0 \\ &= 2((-1 \times 2) - (2 \times 9)) \\ &\quad - 10((3 \times 2) - (2 \times 5)) \\ &= 0 \end{aligned}$$

Solution to Exercise C38

The cofactor A_{13} is $(-1)^{1+3} = (-1)^4 = 1$ times the determinant of the submatrix obtained by removing the top row and third column of $\mathbf{A}$:

$$A_{13} = \begin{vmatrix} 2 & 3 & 5 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 5 & 2 & 3 \\ 5 & 1 & 3 & 4 \end{vmatrix}.$$

The cofactor A_{45} is $(-1)^{4+5} = (-1)^9 = -1$ times the determinant of the submatrix obtained by removing the fourth row and fifth column of $\mathbf{A}$:

$$A_{45} = -\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 \\ 5 & 1 & 2 & 3 \end{vmatrix}.$$

Solution to Exercise C39

We apply Strategy C5:

$$\begin{vmatrix} 0 & 2 & 1 & -1 \\ -3 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 4 & 2 & 0 \end{vmatrix}$$
$$\begin{aligned}
&= 0 - 2\begin{vmatrix} -3 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{vmatrix} + \begin{vmatrix} -3 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 4 & 0 \end{vmatrix} \\
&\qquad - (-1)\begin{vmatrix} -3 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 4 & 2 \end{vmatrix} \\
&= -2\left(-3\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} - 0 + (-1)\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}\right) \\
&\qquad + \left(-3\begin{vmatrix} 0 & 0 \\ 4 & 0 \end{vmatrix} - 0 + (-1)\begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix}\right) \\
&\qquad + \left(-3\begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 0 + 0\right) \\
&= (-2)(-2) + (-1)4 + (-3)(-4) \\
&= 12.
\end{aligned}$$

Solution to Exercise C40

Here,

$$\begin{aligned}
\det \mathbf{A} &= \begin{vmatrix} -3 & 1 \\ 2 & -4 \end{vmatrix} \\
&= (-3 \times (-4)) - (1 \times 2) = 10, \\
\det \mathbf{B} &= \begin{vmatrix} 1 & 1 \\ -2 & 5 \end{vmatrix} \\
&= (1 \times 5) - (1 \times (-2)) = 7
\end{aligned}$$

and

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} -2 & 2 \\ 0 & 1 \end{pmatrix},$$

so

$$\det(\mathbf{A} + \mathbf{B}) = (-2 \times 1) - (2 \times 0) = -2.$$

We have $\det \mathbf{A} + \det \mathbf{B} = 10 + 7 = 17$, and so $\det(\mathbf{A} + \mathbf{B})$ is not equal to $\det \mathbf{A} + \det \mathbf{B}$.

$$\mathbf{AB} = \begin{pmatrix} -5 & 2 \\ 10 & -18 \end{pmatrix},$$

so

$$\det(\mathbf{AB}) = (-5 \times (-18)) - (2 \times 10) = 70.$$

We have $(\det \mathbf{A})(\det \mathbf{B}) = 10 \times 7 = 70$, and so

$$\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B}).$$

Solution to Exercise C41

(a) We apply Strategy C5:

$$\begin{aligned}
\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} &= 0 - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 0 \\
&= -1.
\end{aligned}$$

(b) We apply Strategy C5:

$$\begin{aligned}
\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{vmatrix} - 0 + 0 - 0 \\
&= \begin{vmatrix} k & 0 \\ 0 & 1 \end{vmatrix} - 0 + 0 \\
&= (k \times 1) - (0 \times 0) \\
&= k.
\end{aligned}$$

(c) We evaluate the determinant:

$$\begin{aligned}
\begin{vmatrix} 1 & 0 \\ k & 1 \end{vmatrix} &= (1 \times 1) - (0 \times k) \\
&= 1.
\end{aligned}$$

Solution to Exercise C42

First notice that

$$-2\begin{pmatrix} 1 & -2 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 4 & -8 \end{pmatrix},$$

that is, the first and third rows of $\mathbf{A}$ are proportional. Therefore, by Theorem C16,

$$\det \mathbf{A} = 0.$$

Solution to Exercise C43

We interchange the first and third rows, and apply Theorems C14 and C15, giving

$$\det \mathbf{A} = \begin{vmatrix} 10 & 3 & -4 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 6 & 0 & 0 \\ -1 & 2 & 1 & 0 \end{vmatrix} = (-1) \begin{vmatrix} 0 & 6 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 10 & 3 & -4 & 2 \\ -1 & 2 & 1 & 0 \end{vmatrix}.$$

We use Strategy C5 to evaluate this determinant:

$$\begin{aligned} \det \mathbf{A} &= (-1)\left(0 - 6\begin{vmatrix} 0 & 0 & 1 \\ 10 & -4 & 2 \\ -1 & 1 & 0 \end{vmatrix} + 0 - 0\right) \\ &= 6\left(0 - 0 + \begin{vmatrix} 10 & -4 \\ -1 & 1 \end{vmatrix}\right) \\ &= 6\left((10 \times 1) - (-4 \times (-1))\right) \\ &= 36. \end{aligned}$$

Unit C2

Vector spaces

Introduction

In this unit you will meet a mathematical structure that is one of the most important unifying concepts of pure mathematics. It is that of a *vector space*. A vector space consists of a set of elements called *vectors*, and two operations: *addition of vectors* and *multiplication by a scalar*. These *vectors* need not be vectors in the geometric sense given in Book A; instead, they may be a wide range of objects including complex numbers, functions and matrices.

You will first consider properties of $\mathbb{R}^2$ and $\mathbb{R}^3$, and see how these two- and three-dimensional spaces lead not only to n-dimensional space $\mathbb{R}^n$, but also to the formal definition of a vector space. You will meet a variety of quite different vector spaces and study various concepts relating to vector spaces. For example, you will meet the idea of a *subspace* of a vector space, which is a subset of a vector space that is itself a vector space; this is similar to the relationship between subgroups and groups, which you met in Book B.

The theory of vector spaces introduced in this unit will underpin the remaining units of this book.

1 Vector spaces

In Book A you met the plane and three-dimensional space. In this section you will see that properties that you are familiar with in these two- and three-dimensional spaces also hold for other, quite different-looking *spaces*.

1.1 Euclidean spaces

Recall from Unit A1 *Sets, functions and vectors* that $\mathbb{R}^2$ is the set of all ordered pairs of real numbers, and $\mathbb{R}^3$ is the set of all ordered triples of real numbers. You saw that we can interpret these sets as the plane and as three-dimensional space, respectively, in the following two ways. We can interpret their elements first as the coordinates of points with respect to a specified coordinate system, and second as vectors in component form with respect to this coordinate system.

In this way, once axes have been specified, we can consider the elements of $\mathbb{R}^2$ equivalently as ordered pairs, as points in the plane, or as vectors in the plane. And likewise for $\mathbb{R}^3$, we can consider the elements equivalently as ordered triples of real numbers, as points in three-dimensional space or as vectors in three-dimensional space.

Also in Unit A1, you met two operations: addition of vectors and multiplication of a vector by a scalar. These operations are defined on $\mathbb{R}^2$ and $\mathbb{R}^3$ as follows.

Definitions

In $\mathbb{R}^2$, the set of ordered pairs of real numbers, the operations of **addition** and of **multiplication by a scalar** are defined as:

$$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2),$$

$$\alpha(u_1, u_2) = (\alpha u_1, \alpha u_2), \quad \text{where } \alpha \in \mathbb{R}.$$

In $\mathbb{R}^3$, the set of ordered triples of real numbers, the operations of **addition** and of **multiplication by a scalar** are defined as:

$$(u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3),$$

$$\alpha(u_1, u_2, u_3) = (\alpha u_1, \alpha u_2, \alpha u_3), \quad \text{where } \alpha \in \mathbb{R}.$$

It turns out that $\mathbb{R}^2$ and $\mathbb{R}^3$ are particular instances of a class of mathematical structures called *vector spaces*. In this unit you will meet many other examples, and study the properties that are common to all of them.

You are familiar with vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, but there is no reason to stop at $\mathbb{R}^3$ – why not consider $\mathbb{R}^4$, $\mathbb{R}^5$, or even $\mathbb{R}^n$, for larger positive integers n?

Definitions

Let n be a positive integer. An **ordered n-tuple** is a sequence of real numbers $(u_1, u_2, \ldots, u_n)$. The set of all ordered n-tuples is called **n-dimensional space**, and is denoted by $\mathbb{R}^n$.

To highlight the connection between n-dimensional space (for a positive integer n), denoted by $\mathbb{R}^n$, and 2- and 3-dimensional space with geometrical vectors, the space $\mathbb{R}^n$ is often called a **Euclidean space** and its elements $(u_1, u_2, \ldots, u_n)$ are called vectors. For example, $\mathbb{R}^4$ is the four-dimensional Euclidean space of vectors with four components.

Although it is difficult to visualise vectors in spaces with dimension greater than three, it is possible to carry out exactly the same algebraic manipulations with these vectors, and it turns out that these spaces are also *vector spaces*.

Vector addition and scalar multiplication in $\mathbb{R}^n$ are defined as in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definitions

Let

$$\mathbf{u} = (u_1, u_2, \ldots, u_n) \text{ and } \mathbf{v} = (v_1, v_2, \ldots, v_n)$$

be two vectors in $\mathbb{R}^n$. The operations of **addition** and of **multiplication by a scalar** are defined as:

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) \\ &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n), \\ \alpha\mathbf{u} &= (\alpha u_1, \alpha u_2, \ldots, \alpha u_n), \quad \text{where } \alpha \in \mathbb{R}. \end{aligned}$$

Worked Exercise C20

Let $\mathbf{u} = (1, 1, \ldots, 1)$ and $\mathbf{v} = (1, 2, \ldots, n)$ be two vectors in $\mathbb{R}^n$. Form the vectors $\mathbf{u} + \mathbf{v}$ and $2\mathbf{u}$.

Solution

$$\mathbf{u} + \mathbf{v} = (1, 1, \ldots, 1) + (1, 2, \ldots, n) = (2, 3, \ldots, n + 1)$$

$$2\mathbf{u} = 2(1, 1, \ldots, 1) = (2, 2, \ldots, 2)$$

Exercise C44

Let $\mathbf{u} = (1, -1, 2, 0, -3)$ and $\mathbf{v} = (0, 2, -1, 4, 0)$ be two vectors in $\mathbb{R}^5$. Form the vectors $\mathbf{u} + \mathbf{v}$ and $-3\mathbf{u}$.

This method of generalisation (here from $\mathbb{R}^2$ and $\mathbb{R}^3$ to $\mathbb{R}^n$) is common throughout mathematics. We start with spaces like $\mathbb{R}^2$ and $\mathbb{R}^3$ that we can visualise and look at their properties, and then we generalise these properties to spaces that we cannot easily visualise, such as $\mathbb{R}^n$. So we go from particular cases to a general case.

We can go even further, and think of a vector with a never-ending list of components $(v_1, v_2, v_3, \ldots)$. This is hard to visualise, but is not difficult to handle mathematically. The set of such vectors is called $\mathbb{R}^\infty$, and is an infinite-dimensional vector space. (You will meet a formal definition of *dimension* of a vector space in Section 3.) Vector addition and scalar multiplication are again performed component-wise.

Worked Exercise C21

Let $\mathbf{u} = (1, 0, 1, 0, 1, \ldots)$ and $\mathbf{v} = (1, -2, 3, -4, 5, \ldots)$ be two vectors in $\mathbb{R}^\infty$. Form the vectors $\mathbf{u} + \mathbf{v}$ and $5\mathbf{u}$.

Solution

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (1, 0, 1, 0, 1, \ldots) + (1, -2, 3, -4, 5, \ldots)\\ &= (2, -2, 4, -4, 6, \ldots)\end{aligned}$$

$$5\mathbf{u} = 5(1, 0, 1, 0, 1, \ldots) = (5, 0, 5, 0, 5, \ldots)$$

1.2 Real vector spaces

Before meeting the definition of a vector space, we will look at $\mathbb{R}^4$ and a set of polynomials, and will observe that, despite their apparent differences, these sets share many important properties.

The space $\mathbb{R}^4$

A vector in $\mathbb{R}^4$ has the form (v_1, v_2, v_3, v_4), where v_1, v_2, v_3 and v_4 are real numbers, and the operations of vector addition and scalar multiplication are as defined in the previous subsection.

If we have two vectors $\mathbf{u} = (u_1, u_2, u_3, u_4)$ and $\mathbf{v} = (v_1, v_2, v_3, v_4)$ in $\mathbb{R}^4$, then their sum is

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_1, u_2, u_3, u_4) + (v_1, v_2, v_3, v_4)\\ &= (u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4).\end{aligned}$$

This last vector also belongs to $\mathbb{R}^4$ because each of the four components is a real number, so $\mathbb{R}^4$ is *closed under vector addition*; that is, the closure property (A1), which you met in Unit A2 *Number systems*, holds for the addition of vectors in $\mathbb{R}^4$.

For example, if $\mathbf{u} = (1, 3, 5, 7)$ and $\mathbf{v} = (2, -1, -5, 6)$ are vectors in $\mathbb{R}^4$, then

$$\mathbf{u} + \mathbf{v} = (1, 3, 5, 7) + (2, -1, -5, 6) = (3, 2, 0, 13),$$

which is a vector in $\mathbb{R}^4$.

In fact addition of vectors in $\mathbb{R}^4$ satisfies all the usual rules of arithmetic, as follows. The next worked exercise proves the commutative property (A5) and the additive identity property (A3), and you are asked to prove the remaining two properties in the following exercise.

Addition of vectors in $\mathbb{R}^4$

A1 Closure For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^4$,

$$\mathbf{u} + \mathbf{v} \in \mathbb{R}^4.$$

A2 Associativity For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^4$,

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

A3 Additive identity For all $\mathbf{v} \in \mathbb{R}^4$, and $\mathbf{0} \in \mathbb{R}^4$,

$$\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}.$$

A4 Additive inverses For each $\mathbf{v} \in \mathbb{R}^4$, there is a vector $-\mathbf{v} \in \mathbb{R}^4$ such that

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0} = -\mathbf{v} + \mathbf{v}.$$

A5 Commutativity For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^4$,

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

Worked Exercise C22

Prove that the following properties hold for vector addition in $\mathbb{R}^4$.

(a) The commutative property (A5): $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

(b) The additive identity property (A3): $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$, where $\mathbf{0}$ is the zero vector $(0, 0, 0, 0)$.

Solution

Let $\mathbf{u} = (u_1, u_2, u_3, u_4)$ and $\mathbf{v} = (v_1, v_2, v_3, v_4)$.

(a)
$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_1, u_2, u_3, u_4) + (v_1, v_2, v_3, v_4)\\ &= (u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4)\\ &= (v_1 + u_1, v_2 + u_2, v_3 + u_3, v_4 + u_4)\\ &= (v_1, v_2, v_3, v_4) + (u_1, u_2, u_3, u_4)\\ &= \mathbf{v} + \mathbf{u}\end{aligned}$$

Therefore the commutative property (A5) holds.

(b)
$$\begin{aligned}\mathbf{v} + \mathbf{0} &= (v_1, v_2, v_3, v_4) + (0, 0, 0, 0)\\ &= (v_1 + 0, v_2 + 0, v_3 + 0, v_4 + 0)\\ &= (v_1, v_2, v_3, v_4) = \mathbf{v}\end{aligned}$$

Also, using the commutative property (A5) from part (a) we have

$$\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v},$$

so the additive identity property (A3) holds.

Exercise C45

Prove that the following properties hold for vector addition in $\mathbb{R}^4$.

(a) The associative property (A2): $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.

(b) The additive inverses property (A4): $\mathbf{v}+(-\mathbf{v})=\mathbf{0}=-\mathbf{v}+\mathbf{v}$, where $\mathbf{v}=(v_1,v_2,v_3,v_4)$ and $-\mathbf{v}=(-v_1,-v_2,-v_3,-v_4)$.

Recall from Unit B1 *Symmetry* that a set with a binary operation is a **group** if the following four axioms hold:

G1 (closure); G2 (associativity); G3 (identity) and G4 (inverses).

The first four properties (A1–A4) of vector addition in $\mathbb{R}^4$ show that the set $\mathbb{R}^4$ under the operation of vector addition satisfies these four properties; that is, $(\mathbb{R}^4,+)$ is a group with additive identity the zero vector $(0,0,0,0)$, and $-\mathbf{v}$ the additive inverse of $\mathbf{v}$. The final property, commutativity (A5), shows that it is in fact an abelian group.

These properties all involve vector addition, but $\mathbb{R}^4$ also has some properties that involve scalar multiplication.

Let $\mathbf{v}=(v_1,v_2,v_3,v_4)\in\mathbb{R}^4$ and $\alpha\in\mathbb{R}$. Then

$$\alpha\mathbf{v}=\alpha(v_1,v_2,v_3,v_4)=(\alpha v_1,\alpha v_2,\alpha v_3,\alpha v_4).$$

This vector also belongs to $\mathbb{R}^4$, so $\mathbb{R}^4$ is *closed under scalar multiplication*.

For example, if $\mathbf{v}=(1,2,-5,-3)\in\mathbb{R}^4$ and $\alpha=4$, then

$$\alpha\mathbf{v}=4(1,2,-5,-3)=(4,8,-20,-12),$$

which belongs to $\mathbb{R}^4$.

Note that if you multiply a vector in $\mathbb{R}^4$ by $\beta\in\mathbb{R}$, and then by $\alpha\in\mathbb{R}$, you obtain the same result as multiplying by $\alpha\beta$. This is because, for all $\alpha,\beta\in\mathbb{R}$ and $\mathbf{v}=(v_1,v_2,v_3,v_4)\in\mathbb{R}^4$,

$$\begin{aligned}\alpha(\beta\mathbf{v})&=\alpha(\beta(v_1,v_2,v_3,v_4))\\&=\alpha(\beta v_1,\beta v_2,\beta v_3,\beta v_4)\\&=(\alpha\beta v_1,\alpha\beta v_2,\alpha\beta v_3,\alpha\beta v_4)\\&=(\alpha\beta)(v_1,v_2,v_3,v_4)\\&=(\alpha\beta)\mathbf{v}.\end{aligned}$$

For example, if $\mathbf{v}=(1,2,-5,-3)\in\mathbb{R}^4$ and $\alpha=4$, $\beta=-2$, then

$$\begin{aligned}\alpha(\beta\mathbf{v})&=4(-2(1,2,-5,-3))\\&=4(-2,-4,10,6)\\&=(-8,-16,40,24)\\&=(-8)(1,2,-5,-3)\\&=(\alpha\beta)\mathbf{v}.\end{aligned}$$

Also, if $\mathbf{v}=(v_1,v_2,v_3,v_4)$, then

$$1\mathbf{v}=1(v_1,v_2,v_3,v_4)=(v_1,v_2,v_3,v_4)=\mathbf{v}.$$

These properties of scalar multiplication of vectors in $\mathbb{R}^4$ can be summarised as follows.

Scalar multiplication of vectors in $\mathbb{R}^4$

S1 Closure For all $\mathbf{v} \in \mathbb{R}^4$, and $\alpha \in \mathbb{R}$,

$$\alpha\mathbf{v} \in \mathbb{R}^4.$$

S2 Associativity For all $\mathbf{v} \in \mathbb{R}^4$, and $\alpha, \beta \in \mathbb{R}$,

$$\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}.$$

S3 Scalar multiplicative identity For all $\mathbf{v} \in \mathbb{R}^4$,

$$1\mathbf{v} = \mathbf{v}.$$

Finally, there are two distributive properties that connect vector addition and scalar multiplication.

For example, if $\mathbf{u} = (1, 3, 5, 7)$ and $\mathbf{v} = (2, -1, -5, 6)$ are vectors in $\mathbb{R}^4$, and $\alpha = 3$ and $\beta = 4$, then

$$\begin{aligned}\alpha(\mathbf{u} + \mathbf{v}) &= 3\left((1, 3, 5, 7) + (2, -1, -5, 6)\right)\\ &= 3(3, 2, 0, 13)\\ &= (9, 6, 0, 39)\end{aligned}$$

and

$$\begin{aligned}\alpha\mathbf{u} + \alpha\mathbf{v} &= 3(1, 3, 5, 7) + 3(2, -1, -5, 6)\\ &= (3, 9, 15, 21) + (6, -3, -15, 18)\\ &= (9, 6, 0, 39),\end{aligned}$$

which illustrates the first distributive property. Also,

$$\begin{aligned}(\alpha + \beta)\mathbf{v} &= (3 + 4)(2, -1, -5, 6)\\ &= 7(2, -1, -5, 6)\\ &= (14, -7, -35, 42)\end{aligned}$$

and

$$\begin{aligned}\alpha\mathbf{v} + \beta\mathbf{v} &= 3(2, -1, -5, 6) + 4(2, -1, -5, 6)\\ &= (6, -3, -15, 18) + (8, -4, -20, 24)\\ &= (14, -7, -35, 42),\end{aligned}$$

which illustrates the second.

These properties connecting vector addition and scalar multiplication can be summarised as follows.

Combining addition and scalar multiplication of vectors in $\mathbb{R}^4$

D1 Distributivity For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^4$, and $\alpha \in \mathbb{R}$,

$$\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}.$$

D2 Distributivity For all $\mathbf{v} \in \mathbb{R}^4$, and $\alpha, \beta \in \mathbb{R}$,

$$(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}.$$

The space of quadratic polynomials

Let us now look at another, apparently very different set of elements. This is the set of *quadratic polynomials*, namely, functions of the form

$$\begin{aligned} p : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto a + bx + cx^2, \end{aligned}$$

where $a, b, c \in \mathbb{R}$. We call this set P_3 because it comprises all the real polynomials of degree less than 3. Thus

$$P_3 = \{p(x) : p(x) = a + bx + cx^2,\ a, b, c \in \mathbb{R}\}.$$

Here we have used the convention from Book A that when a real function is specified only by a rule, it is understood that the domain of the function is the set of all real numbers for which the rule is applicable, and the codomain of the function is $\mathbb{R}$.

(We write the terms of the polynomial in increasing order of powers here, as usually done when working within a *vector space* of polynomials.)

To simplify the notation further, we write

$$P_3 = \{a + bx + cx^2 : a, b, c \in \mathbb{R}\}.$$

This set includes the quadratic polynomials (where c is non-zero), the linear polynomials (where c is 0 and b is non-zero) and constants (where b and c are 0 and a is non-zero), as well as the zero polynomial (where $a = b = c = 0$). At first sight, there is no reason why this set of elements should have the properties that we have just shown are satisfied by $\mathbb{R}^4$; however, these properties all hold for this set as well.

First we consider the properties A1–A5 involving addition.

Consider $p_1(x) = a_1 + b_1x + c_1x^2$ and $p_2(x) = a_2 + b_2x + c_2x^2$, then

$$\begin{aligned} p_1(x) + p_2(x) &= (a_1 + b_1x + c_1x^2) + (a_2 + b_2x + c_2x^2) \\ &= (a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2, \end{aligned}$$

which also belongs to P_3. Therefore the closure property (A1) holds for addition in P_3.

For example, $3 + 4x - 2x^2$ and $5 - 3x + 7x^2$ both belong to P_3, and

$$(3 + 4x - 2x^2) + (5 - 3x + 7x^2) = 8 + x + 5x^2,$$

which also belongs to P_3. The next worked exercise proves the commutative property (A5) and the additive inverses property (A4), and you are asked to prove the remaining two properties in the following exercise.

Worked Exercise C23

Prove that the following properties hold for addition in P_3.

(a) The commutative property (A5): $p_1(x) + p_2(x) = p_2(x) + p_1(x)$.

(b) The additive inverses property (A4):
$p_1(x) + (-p_1(x)) = \mathbf{0} = -p_1(x) + p_1(x)$.

Solution

(a)
$$\begin{aligned} p_1(x) + p_2(x) &= (a_1 + b_1x + c_1x^2) + (a_2 + b_2x + c_2x^2) \\ &= (a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2 \\ &= (a_2 + a_1) + (b_2 + b_1)x + (c_2 + c_1)x^2 \\ &= (a_2 + b_2x + c_2x^2) + (a_1 + b_1x + c_1x^2) \\ &= p_2(x) + p_1(x) \end{aligned}$$

Therefore the commutative property (A5) holds for addition in P_3.

(b)
$$\begin{aligned} p_1(x) + (-p_1(x)) &= (a_1 + b_1x + c_1x^2) + (-a_1 - b_1x - c_1x^2) \\ &= (a_1 - a_1) + (b_1 - b_1)x + (c_1 - c_1)x^2 \\ &= 0 + 0x + 0x^2 = \mathbf{0} \end{aligned}$$

Also, using the commutative property (A5) from part (a) we have

$$p_1(x) + (-p_1(x)) = \mathbf{0} = -p_1(x) + p_1(x),$$

so the additive inverses property (A4) holds for addition in P_3.

Exercise C46

Prove that the following properties hold for addition in P_3.

(a) The associative property (A2):
$(p_1(x) + p_2(x)) + p_3(x) = p_1(x) + (p_2(x) + p_3(x))$.

(b) The additive identity property (A3): $p_1(x) + \mathbf{0} = p_1(x) = \mathbf{0} + p_1(x)$, where $\mathbf{0} = 0 + 0x + 0x^2$ is the zero polynomial in P_3.

It follows that P_3 satisfies the same addition properties as $\mathbb{R}^4$, and therefore P_3 is also an abelian group under addition.

We can multiply a polynomial through by a real constant; that is, by a scalar. In fact P_3 has the same properties involving scalar multiplication as $\mathbb{R}^4$.

Let $p(x) = a + bx + cx^2$ and $\alpha \in \mathbb{R}$, then

$$\alpha p(x) = \alpha(a + bx + cx^2) = (\alpha a) + (\alpha b)x + (\alpha c)x^2,$$

which also belongs to P_3. So P_3 is *closed under scalar multiplication*; that is, the closure property (S1) holds for P_3 under scalar multiplication.

In the following exercise you are asked to check the remaining properties involving scalar multiplication (S2 and S3), for a particular case.

Exercise C47

Let $p(x) = 1 - x + 2x^2$ and $\alpha = 2$, $\beta = -3$. Show that the following properties hold for these scalars and this quadratic polynomial.

(a) The identity property (S3): $1 \times p(x) = p(x)$.

(b) The associative property (S2): $\alpha(\beta p(x)) = (\alpha\beta)p(x)$.

To finish looking at the properties of P_3, we note that the distributive properties (D1 and D2) that connect addition and scalar multiplication hold for P_3; the proofs simply involve multiplying out brackets. For all $p_1(x), p_2(x) \in P_3$ and $\alpha, \beta \in \mathbb{R}$,

$$\alpha(p_1(x) + p_2(x)) = \alpha p_1(x) + \alpha p_2(x)$$

and

$$(\alpha + \beta)p_1(x) = \alpha p_1(x) + \beta p_1(x).$$

So $\mathbb{R}^4$ and P_3 satisfy the same set of properties with respect to addition and scalar multiplication, even though $\mathbb{R}^4$ is a Euclidean space and P_3 is a set of polynomials. The idea that connects them is the concept of a *vector space*.

Vector space definition

In Book B we studied symmetries of geometric figures, and then abstracted the properties to obtain the definition of a group. We go through a similar process here. We have just studied $\mathbb{R}^4$ and P_3, and we now abstract from them the definition of a *vector space*. We then go on to look at other examples of vector spaces. The elements of these vector spaces are of diverse types: complex numbers, functions, matrices, and many others.

The definition of a vector space is one of the longest definitions in mathematics. It looks formidable, but the axioms A1–A5, S1–S3 and D1–D2 are precisely the properties we checked for $\mathbb{R}^4$ and P_3. Thus this

definition follows naturally from our previous examples. As for $\mathbb{R}^4$ and P_3, axioms A1–A5 refer to vector addition (implying that a vector space is an abelian group under addition), S1–S3 refer to scalar multiplication, and D1–D2 to how we combine these operations. Therefore a *vector space* is a set of objects called *vectors* that can be added together and scalar multiplied in such a way that all the usual properties of arithmetic hold. Thus the definition includes the properties for addition, the properties for scalar multiplication and the properties of how these two operations combine.

Definition

A **real vector space** consists of a set V of elements called **vectors** and two operations, vector addition and scalar multiplication, such that the following axioms hold.

Axioms for addition

A1 Closure For all $\mathbf{v}_1, \mathbf{v}_2 \in V$,

$$\mathbf{v}_1 + \mathbf{v}_2 \in V.$$

A2 Associativity For all $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$,

$$(\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3).$$

A3 Additive identity For all $\mathbf{v} \in V$, there is a zero element $\mathbf{0} \in V$ satisfying

$$\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}.$$

A4 Additive inverses For each $\mathbf{v} \in V$, there is an element $-\mathbf{v}$ (its additive inverse) such that

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0} = -\mathbf{v} + \mathbf{v}.$$

A5 Commutativity For all $\mathbf{v}_1, \mathbf{v}_2 \in V$,

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1.$$

Axioms A1–A5 imply that $(V, +)$ is an *abelian group*.

Axioms for scalar multiplication

S1 Closure For all $\mathbf{v} \in V$, and $\alpha \in \mathbb{R}$,

$$\alpha\mathbf{v} \in V.$$

S2 Associativity For all $\mathbf{v} \in V$, and $\alpha, \beta \in \mathbb{R}$,

$$\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}.$$

S3 Scalar multiplicative identity For all $\mathbf{v} \in V$,

$$1\mathbf{v} = \mathbf{v}.$$

Axioms combining addition and scalar multiplication

D1 Distributivity For all $\mathbf{v}_1, \mathbf{v}_2 \in V$, and $\alpha \in \mathbb{R}$,

$$\alpha(\mathbf{v}_1 + \mathbf{v}_2) = \alpha\mathbf{v}_1 + \alpha\mathbf{v}_2.$$

D2 Distributivity For all $\mathbf{v} \in V$, and $\alpha, \beta \in \mathbb{R}$,

$$(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}.$$

The word 'real' in this definition refers to the fact that *the scalars used in forming scalar multiples are real numbers*; that is, a real vector space is a vector space over the field $\mathbb{R}$ (which means that the scalars are elements in $\mathbb{R}$). More generally, it is possible to define a vector space over *any* field, so it is also possible to form complex and rational vector spaces, where the vectors are scalar multiplied by complex and rational numbers, respectively. This is because the sets of complex and rational numbers are also fields. However, we are only concerned with real vector spaces in this module.

It is worth noting that $\mathbb{R}$ itself is a real vector space: the fact that the vector space axioms hold for $V = \mathbb{R}$ follows from the field properties that hold for $\mathbb{R}$, which were shown in Unit A2 when considering the arithmetic of real numbers.

Where we use the term *vector* for the elements of vector spaces, many mathematical texts use the terms *element* and *vector* interchangeably.

Checking the axioms

We now look at the set $V = \{a\cos x + b\sin x : a, b \in \mathbb{R}\}$ of functions, and show that it is a real vector space by checking all the axioms in the definition. You will not be asked to check *all* these axioms in a single exercise: this example simply illustrates how it can be done.

Addition and scalar multiplication are defined on V as follows.

If $a_1\cos x + b_1\sin x$ and $a_2\cos x + b_2\sin x$ are vectors of V, and $\alpha \in \mathbb{R}$, then

$$\begin{aligned}(a_1\cos x + b_1\sin x) &+ (a_2\cos x + b_2\sin x)\\ &= (a_1 + a_2)\cos x + (b_1 + b_2)\sin x\end{aligned}$$

and

$$\alpha(a_1\cos x + b_1\sin x) = \alpha a_1\cos x + \alpha b_1\sin x.$$

For example,

$$(3\cos x + 2\sin x) + (4\cos x - 6\sin x) = 7\cos x - 4\sin x$$

and

$$-5(3\cos x + 4\sin x) = -15\cos x - 20\sin x.$$

We check the axioms one by one.

A1 Closure V is closed under addition of functions, since, if $a_1 \cos x + b_1 \sin x$ and $a_2 \cos x + b_2 \sin x$ are vectors of V, then

$$\begin{aligned}(a_1 \cos x + b_1 \sin x) + (a_2 \cos x + b_2 \sin x)\\ = (a_1 + a_2)\cos x + (b_1 + b_2)\sin x,\end{aligned}$$

which is a vector of V.

A2 Associativity Addition is associative, since, if $a_1 \cos x + b_1 \sin x$, $a_2 \cos x + b_2 \sin x$ and $a_3 \cos x + b_3 \sin x$ are vectors of V, then

$$\begin{aligned}&((a_1 \cos x + b_1 \sin x) + (a_2 \cos x + b_2 \sin x)) + (a_3 \cos x + b_3 \sin x)\\ &\quad = ((a_1 + a_2)\cos x + (b_1 + b_2)\sin x) + (a_3 \cos x + b_3 \sin x)\\ &\quad = (a_1 + a_2 + a_3)\cos x + (b_1 + b_2 + b_3)\sin x\end{aligned}$$

and

$$\begin{aligned}&(a_1 \cos x + b_1 \sin x) + ((a_2 \cos x + b_2 \sin x) + (a_3 \cos x + b_3 \sin x))\\ &\quad = (a_1 \cos x + b_1 \sin x) + ((a_2 + a_3)\cos x + (b_2 + b_3)\sin x)\\ &\quad = (a_1 + a_2 + a_3)\cos x + (b_1 + b_2 + b_3)\sin x.\end{aligned}$$

A3 Additive identity The zero vector is $0 \cos x + 0 \sin x$, since this is in V and, if $a \cos x + b \sin x \in V$, then

$$(a \cos x + b \sin x) + (0 \cos x + 0 \sin x) = a \cos x + b \sin x$$

and

$$(0 \cos x + 0 \sin x) + (a \cos x + b \sin x) = a \cos x + b \sin x.$$

A4 Additive inverses The additive inverse of $a \cos x + b \sin x$ is $-a \cos x - b \sin x$, since this is in V and, if $a \cos x + b \sin x \in V$, then

$$(a \cos x + b \sin x) + (-a \cos x - b \sin x) = 0 \cos x + 0 \sin x$$

and

$$(-a \cos x - b \sin x) + (a \cos x + b \sin x) = 0 \cos x + 0 \sin x.$$

A5 Commutativity Addition is commutative, since, if $a_1 \cos x + b_1 \sin x$ and $a_2 \cos x + b_2 \sin x$ are vectors of V, then

$$\begin{aligned}(a_1 \cos x + b_1 \sin x) + (a_2 \cos x + b_2 \sin x)\\ = (a_1 + a_2)\cos x + (b_1 + b_2)\sin x\end{aligned}$$

and

$$\begin{aligned}(a_2 \cos x + b_2 \sin x) + (a_1 \cos x + b_1 \sin x)\\ = (a_2 + a_1)\cos x + (b_2 + b_1)\sin x\\ = (a_1 + a_2)\cos x + (b_1 + b_2)\sin x.\end{aligned}$$

S1 Closure V is closed under scalar multiplication, since, for $a \cos x + b \sin x \in V$ and $\alpha \in \mathbb{R}$, we have

$$\alpha(a \cos x + b \sin x) = \alpha a \cos x + \alpha b \sin x.$$

This is in V, since $\alpha a, \alpha b \in \mathbb{R}$.

S2 Associativity For $\alpha, \beta \in \mathbb{R}$ and $a\cos x + b\sin x \in V$, we have

$$\begin{aligned}\alpha\left(\beta(a\cos x + b\sin x)\right) &= \alpha(\beta a\cos x + \beta b\sin x)\\ &= \alpha\beta a\cos x + \alpha\beta b\sin x\\ &= (\alpha\beta)(a\cos x + b\sin x).\end{aligned}$$

S3 Scalar multiplicative identity For $a\cos x + b\sin x \in V$, we have

$$1(a\cos x + b\sin x) = a\cos x + b\sin x.$$

D1 Distributivity For $\alpha \in \mathbb{R}$ and $a_1\cos x + b_1\sin x$ and $a_2\cos x + b_2\sin x$ in V, we have

$$\begin{aligned}&\alpha((a_1\cos x + b_1\sin x) + (a_2\cos x + b_2\sin x))\\ &\qquad= \alpha\left((a_1+a_2)\cos x + (b_1+b_2)\sin x\right)\\ &\qquad= \alpha(a_1+a_2)\cos x + \alpha(b_1+b_2)\sin x\end{aligned}$$

and

$$\begin{aligned}&\alpha(a_1\cos x + b_1\sin x) + \alpha(a_2\cos x + b_2\sin x)\\ &\qquad= \alpha a_1\cos x + \alpha b_1\sin x + \alpha a_2\cos x + \alpha b_2\sin x\\ &\qquad= \alpha(a_1+a_2)\cos x + \alpha(b_1+b_2)\sin x.\end{aligned}$$

D2 Distributivity For $\alpha, \beta \in \mathbb{R}$ and $a\cos x + b\sin x \in V$, we have

$$\begin{aligned}&(\alpha+\beta)(a\cos x + b\sin x)\\ &\qquad= (\alpha+\beta)a\cos x + (\alpha+\beta)b\sin x\\ &\qquad= \alpha a\cos x + \alpha b\sin x + \beta a\cos x + \beta b\sin x\end{aligned}$$

and

$$\begin{aligned}&\alpha(a\cos x + b\sin x) + \beta(a\cos x + b\sin x)\\ &\qquad= \alpha a\cos x + \alpha b\sin x + \beta a\cos x + \beta b\sin x.\end{aligned}$$

Since all the vector space properties are satisfied, V is a vector space.

We now look briefly at some further examples of vector spaces, to give you some idea of the different areas of mathematics in which this concept arises.

The set of linear polynomials P_2

The set P_2 of linear polynomials comprises the real polynomials of degree less than 2; that is, the polynomials of the form $p(x) = a + bx$, where $a, b \in \mathbb{R}$. Vector addition and scalar multiplication are defined on P_2 as follows.

If $p(x) = a + bx$ and $q(x) = c + dx$, and $\alpha \in \mathbb{R}$, then

$$p(x) + q(x) = (a + bx) + (c + dx) = (a + c) + (b + d)x$$

and

$$\alpha p(x) = \alpha(a + bx) = (\alpha a) + (\alpha b)x.$$

The result of each of these operations is a linear polynomial, so P_2 is closed under the operations of addition and scalar multiplication, and therefore satisfies the closure axioms (A1 and S1). The other axioms can be checked in the same way.

More generally, for each positive integer n, the set P_n of real polynomials of degree less than n, with the usual operations of addition and scalar multiplication, is a vector space.

The set of complex numbers $\mathbb{C}$

The set $\mathbb{C}$ comprises the numbers of the form $a + bi$, where $i^2 = -1$ and $a, b \in \mathbb{R}$. Vector addition and scalar multiplication are defined on $\mathbb{C}$ as

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and

$$\alpha(a + bi) = (\alpha a) + (\alpha b)i.$$

This is a *real* vector space because we multiply the *complex* number (the vector) by a *real* number (the scalar).

The result of each of these operations is a complex number, so $\mathbb{C}$ is closed under the operations of vector addition and scalar multiplication, and therefore satisfies the closure axioms (A1 and S1). The other axioms can be checked in the same way.

The set $M_{2,3}$ of 2×3 matrices with real entries

The set $M_{2,3}$ comprises the 2×3 matrices of the form

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}, \quad \text{where } a, b, c, d, e, f \in \mathbb{R}.$$

Vector addition and scalar multiplication are defined on $M_{2,3}$ as follows.

If $\mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \end{pmatrix}$, and $\alpha \in \mathbb{R}$, then

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \\ a_4 + b_4 & a_5 + b_5 & a_6 + b_6 \end{pmatrix} \end{aligned}$$

and

$$\alpha\mathbf{A} = \alpha \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} = \begin{pmatrix} \alpha a_1 & \alpha a_2 & \alpha a_3 \\ \alpha a_4 & \alpha a_5 & \alpha a_6 \end{pmatrix}.$$

The result of each of these operations is a 2×3 matrix with real entries, so this set is closed under the operations of vector addition and scalar multiplication, and therefore satisfies the closure axioms (A1 and S1). The other axioms can be checked in the same way.

More generally, for positive integers m and n, the set $M_{m,n}$ of $m \times n$ matrices with real entries is a vector space under the operations of vector addition and scalar multiplication.

The set $\mathbb{R}^\infty$

If $\mathbf{u} = (u_1, u_2, \ldots)$ and $\mathbf{v} = (v_1, v_2, \ldots)$ belong to $\mathbb{R}^\infty$, and $\alpha \in \mathbb{R}$, then

$$\mathbf{u} + \mathbf{v} = (u_1, u_2, \ldots) + (v_1, v_2, \ldots) = (u_1 + v_1, u_2 + v_2, \ldots)$$

and

$$\alpha\mathbf{u} = \alpha(u_1, u_2, \ldots) = (\alpha u_1, \alpha u_2, \ldots).$$

The result of each of these operations is a vector of $\mathbb{R}^\infty$, so $\mathbb{R}^\infty$ is closed under the operations of vector addition and scalar multiplication, and therefore satisfies the closure axioms (A1 and S1). The other axioms can be checked in the same way.

These examples are only a few of the many real vector spaces. You will meet more of them as you work through this unit, and as you encounter other mathematical concepts in the remainder of this module.

We finish this section by looking at some sets that are not vector spaces. In each case you should assume the usual definitions of addition and scalar multiplication for the elements of these sets to show that these sets are not vector spaces.

Worked Exercise C24

Show that neither of the following sets is a real vector space.

(a) $V = \{\text{all polynomials of degree equal to 5}\}$

(b) $V = \{a + bi \in \mathbb{C} : a \geq 0\}$

Solution

(a) Recall that P_3 is the set of all polynomials of degree less than 3. Here we have all polynomials of degree *equal* to 5. Therefore the closure axiom (A1) is a good place to start. All that is needed is one pair of polynomials in V whose sum is not in V.

Consider the polynomials $p(x) = x + x^5$, $q(x) = x - x^5$, both of degree 5. We have $p(x) + q(x) = (x + x^5) + (x - x^5) = 2x$, which is a polynomial of degree 1 and so not in V.

Therefore addition on the set of all polynomials of degree equal to 5 fails to satisfy the closure axiom (A1), so V is not a real vector space.

Other axioms also fail, or do not make sense; for example, V fails the additive identity axiom (A3) since it contains no zero vector, and therefore the additive inverses property (A4) makes no sense here.

(b) We know that the set of *all* complex numbers is a vector space, so the condition $a \geq 0$ is important here. The closure axiom (S1) is a good place to start. Can we find a complex number in V where scalar multiplication by, say, $\alpha = -1$ is not in V?

Consider $z = 1$ in V, and let $\alpha = -1$, then $\alpha z = -1$, which is not of the form $a + bi$ where $a \geq 0$.

Therefore scalar multiplication on the set of complex numbers of the form $a + bi$, where $a \geq 0$, fails to satisfy the closure axiom (S1), so V is not a real vector space.

Other axioms also fail; for example, V fails the additive inverses axiom (A4) since $z = 1$ has no additive inverse in V. In addition, because axiom S1 fails, the axioms S2, D1 and D2 are meaningless.

Exercise C48

Show that neither of the following sets is a real vector space.

(a) $V = \{(x, y) \in \mathbb{R}^2 : y = 2x + 1\}$

(b) $V = \left\{ \begin{pmatrix} 0 & a \\ b & c \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$

2 Linear combinations and spanning sets

In this section you will see that in a vector space, some sets of vectors are special. These special sets are such that every other vector in the space can be produced by adding combinations and scalar multiples of vectors just in this special set.

2.1 Linear combinations

We begin by looking at the different ways in which we can express a single vector in $\mathbb{R}^2$ as a combination of two other vectors.

For example, the vector $(5, 3)$ in $\mathbb{R}^2$, illustrated in Figure 1, can be written as

$$(5, 3) = 5(1, 0) + 3(0, 1).$$

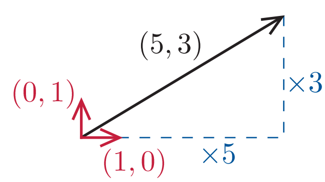


Figure 1 The vector $(5, 3)$ as a linear combination of $(1, 0)$ and $(0, 1)$

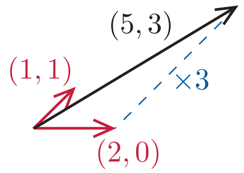


Figure 2 The vector $(5, 3)$ as a linear combination of $(2, 0)$ and $(1, 1)$

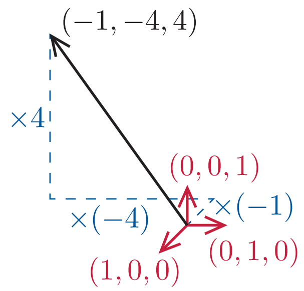


Figure 3 A vector in $\mathbb{R}^3$ as a linear combination of three vectors

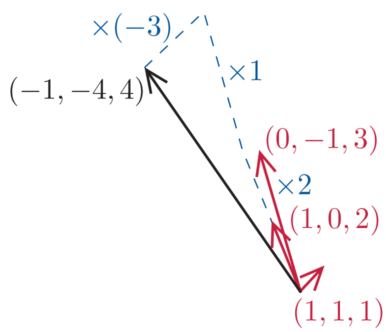


Figure 4 A vector in $\mathbb{R}^3$ as a linear combination of three vectors

We could also write $(5, 3)$ in terms of $(2, 0)$ and $(1, 1)$, illustrated in Figure 2. In this case we have

$$(5, 3) = 1(2, 0) + 3(1, 1).$$

If you look at the right-hand sides of these equations, you will see that they both have the same form. In each case we have written

$$(5, 3) = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2,$$

where $\mathbf{v}_1 = (1, 0)$, $\mathbf{v}_2 = (0, 1)$, $\alpha = 5$ and $\beta = 3$ in the first case, and $\mathbf{v}_1 = (2, 0)$, $\mathbf{v}_2 = (1, 1)$, $\alpha = 1$ and $\beta = 3$ in the second case.

We call $\alpha \mathbf{v}_1 + \beta \mathbf{v}_2$ a *linear combination* of the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

Because $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors in $\mathbb{R}^2$, so are $\alpha \mathbf{v}_1$ and $\beta \mathbf{v}_2$, since they are scalar multiples of $\mathbf{v}_1$ and $\mathbf{v}_2$; and hence so is $\alpha \mathbf{v}_1 + \beta \mathbf{v}_2$, since it is the sum of two vectors in $\mathbb{R}^2$. So $\alpha \mathbf{v}_1 + \beta \mathbf{v}_2$ is also a vector in $\mathbb{R}^2$.

Similarly in $\mathbb{R}^3$, the vector $(-1, -4, 4)$, illustrated in Figure 3, can be written as

$$(-1, -4, 4) = -1(1, 0, 0) - 4(0, 1, 0) + 4(0, 0, 1)$$

or as illustrated in Figure 4, in terms of the three vectors $(1, 0, 2)$, $(0, -1, 3)$ and $(1, 1, 1)$ as

$$(-1, -4, 4) = 2(1, 0, 2) + 1(0, -1, 3) - 3(1, 1, 1).$$

These are two examples: they are not the only possibilities. Each of these equations has the form

$$(-1, -4, 4) = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 + \gamma \mathbf{v}_3,$$

where the expression on the right-hand side of the equation is a *linear combination* of three vectors.

These linear combinations of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ are particular examples of the following definition.

Definition

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ belong to a vector space V. Then a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ is a vector of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k,$$

where $\alpha_1, \alpha_2, \dots, \alpha_k$ are real numbers. This vector also belongs to V.

We begin by looking at how we can form linear combinations of vectors, and then investigate whether we can write a particular vector as a linear combination of other vectors in the same vector space.

In the worked exercises and exercises of this section we have tried to keep the arithmetic simple by using *integer* scalar multiples and coordinates. In general, any real numbers may occur.

Worked Exercise C25

(a) In $\mathbb{R}^3$, calculate the linear combination $2\mathbf{v}_1 + 3\mathbf{v}_2$ when $\mathbf{v}_1 = (1, 0, 3)$ and $\mathbf{v}_2 = (0, 2, -1)$.

(b) In $\mathbb{R}^4$, calculate the linear combination $2\mathbf{v}_1 + 3\mathbf{v}_2 + 4\mathbf{v}_3 - \mathbf{v}_4$ when $\mathbf{v}_1 = (1, 0, 3, 1)$, $\mathbf{v}_2 = (0, 2, 0, -1)$, $\mathbf{v}_3 = (0, 1, -2, 0)$ and $\mathbf{v}_4 = (2, 10, -2, -1)$.

Solution

(a)
$$\begin{aligned} 2\mathbf{v}_1 + 3\mathbf{v}_2 &= 2(1, 0, 3) + 3(0, 2, -1) \\ &= (2, 0, 6) + (0, 6, -3) = (2, 6, 3) \end{aligned}$$

(b)
$$\begin{aligned} &2\mathbf{v}_1 + 3\mathbf{v}_2 + 4\mathbf{v}_3 - \mathbf{v}_4 \\ &\quad = 2(1, 0, 3, 1) + 3(0, 2, 0, -1) + 4(0, 1, -2, 0) - (2, 10, -2, -1) \\ &\quad = (2, 0, 6, 2) + (0, 6, 0, -3) + (0, 4, -8, 0) - (2, 10, -2, -1) \\ &\quad = (0, 0, 0, 0) \end{aligned}$$

Exercise C49

(a) In $\mathbb{R}^2$, let $\mathbf{v}_1 = (0, 3)$ and $\mathbf{v}_2 = (2, 1)$. Calculate the linear combination $4\mathbf{v}_1 - 2\mathbf{v}_2$.

(b) In $\mathbb{R}^4$, let $\mathbf{v}_1 = (1, 2, 1, 3)$ and $\mathbf{v}_2 = (2, 1, 0, -1)$. Calculate the linear combination $3\mathbf{v}_1 + 2\mathbf{v}_2$.

We now look at linear combinations of vectors in vector spaces other than $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^4$. In the worked exercise and exercise that follow, we assume that the operations of vector addition and scalar multiplication for polynomials, matrices and functions are the usual ones.

Worked Exercise C26

For each of the following vector spaces V and vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ in V, form the linear combination $3\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3$.

(a) $V = P_3$, $\quad \mathbf{v}_1 = 1 + x + x^2$, $\quad \mathbf{v}_2 = 1 - x$, $\quad \mathbf{v}_3 = x + x^2$.

(b) $V = M_{2,3}$, $\quad \mathbf{v}_1 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{pmatrix}$, $\quad \mathbf{v}_2 = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3 & -4 \end{pmatrix}$,
$\mathbf{v}_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$.

Solution

(a) $$\begin{aligned} 3\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 &= 3(1 + x + x^2) - 2(1 - x) + (x + x^2) \\ &= 1 + 6x + 4x^2 \end{aligned}$$

(b) $3\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3$

$$\begin{aligned} &= 3\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{pmatrix} - 2\begin{pmatrix} 2 & -1 & 0 \\ 0 & 3 & -4 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 2 & 6 \\ 0 & -7 & 18 \end{pmatrix} \end{aligned}$$

Exercise C50

For each of the following vector spaces V and vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V, form the linear combination $2\mathbf{v}_1 - 4\mathbf{v}_2$.

(a) $V = P_3, \quad \mathbf{v}_1 = 2 - x + 3x^2, \quad \mathbf{v}_2 = -1 + x.$

(b) V is the set of all real functions, $\quad \mathbf{v}_1 = \sin x, \quad \mathbf{v}_2 = x\cos x.$

(c) $V = M_{2,2}, \quad \mathbf{v}_1 = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}.$

Now that we have formed linear combinations of different numbers of vectors in various vector spaces, we consider the harder problem of deciding whether we can express a given vector as a linear combination of a particular set of vectors. In the next worked exercise, we look at an example before giving a general strategy.

Worked Exercise C27

Determine whether $(3, -1)$ can be expressed as a linear combination of each of the following.

(a) $\mathbf{v}_1 = (2, 0)$ and $\mathbf{v}_2 = (1, 1)$. (b) $\mathbf{v}_1 = (2, 2)$ and $\mathbf{v}_2 = (1, 1)$.

(c) $\mathbf{v}_1 = (9, -3)$ and $\mathbf{v}_2 = (-6, 2)$.

Solution

(a) We need to find real numbers α and β such that

$$(3,-1) = \alpha(2,0) + \beta(1,1),$$

that is,

$$(3,-1) = (2\alpha + \beta, \beta).$$

We equate the two first coordinates (components) to get $3 = 2\alpha + \beta$, and then the two second coordinates (components) to get $-1 = \beta$.

Equating corresponding coordinates, we obtain the system of linear equations

$$\begin{aligned} 2\alpha + \beta &= 3 \\ \beta &= -1. \end{aligned}$$

Substituting $\beta = -1$ in the first equation gives $\alpha = 2$. So

$$\begin{aligned} (3,-1) &= 2(2,0) - 1(1,1) \\ &= 2\mathbf{v}_1 - \mathbf{v}_2. \end{aligned}$$

(b) We need to find real numbers α and β such that

$$(3,-1) = \alpha(2,2) + \beta(1,1),$$

that is,

$$(3,-1) = (2\alpha + \beta, 2\alpha + \beta).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} 2\alpha + \beta &= 3 \\ 2\alpha + \beta &= -1. \end{aligned}$$

The left-hand sides of these equations are the same but the right-hand sides are different, so we can immediately conclude that they are inconsistent. Alternatively, subtracting the second equation from the first yields the equation $0 = 4$.

This pair of equations is inconsistent, since no values of α and β satisfy both of them.

We might have expected this since any linear combination of $(1,1)$ and $(2,2)$ must have both coordinates the same.

We cannot express $(3,-1)$ as a linear combination of these two vectors.

(c) We need to find real numbers α and β such that

$$(3,-1) = \alpha(9,-3) + \beta(-6,2),$$

that is,

$$(3,-1) = (9\alpha - 6\beta, -3\alpha + 2\beta).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} 9\alpha - 6\beta &= 3 \\ -3\alpha + 2\beta &= -1. \end{aligned}$$

Multiplying the second equation by -3 yields the first equation.

These equations are equivalent to the single equation

$$3\alpha - 2\beta = 1,$$

so any values of α and β that satisfy this equation give a solution. Thus in this case there are infinitely many solutions. For example, if $\alpha = 1$, then $\beta = 1$, and

$$(3, -1) = (9, -3) + (-6, 2) = \mathbf{v}_1 + \mathbf{v}_2.$$

Other solutions include $\alpha = \frac{1}{3}$, $\beta = 0$ and $\alpha = 0$, $\beta = -\frac{1}{2}$. Here there are infinitely many solutions because both $(9, -3)$ and $(-6, 2)$ are multiples of $(3, -1)$.

The following strategy describes the method we have just used.

Strategy C6

To determine whether a given vector $\mathbf{v}$ can be written as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$:

1. write $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k$
2. use this expression to write down a system of linear equations in the unknowns $\alpha_1, \alpha_2, \ldots, \alpha_k$
3. solve the resulting system of equations, if possible.

Then $\mathbf{v}$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if and only if the system has a solution.

Recall from Unit C1 *Linear equations and matrices* that a system of linear equations may have no solution, a unique solution, or infinitely many solutions. Therefore this strategy may give no solution, a unique solution, or infinitely many solutions, as we saw in Worked Exercise C27.

When dealing with polynomial functions, such as those in P_3, we use the fact that two polynomial equations in the variable x are equal if and only if the coefficients of corresponding powers of x are equal, and *equate corresponding coefficients*.

Worked Exercise C28

(a) In $\mathbb{R}^3$, express the vector $(1,1,1)$ as a linear combination of the vectors $(1,0,1)$, $(0,1,2)$ and $(-1,1,0)$.

(b) In P_3, express the polynomial $2+2x+5x^2$ as a linear combination of the polynomials $1+3x^2$ and $2x-x^2$.

Solution

We follow the steps of Strategy C6.

(a) Let α, β and γ be real numbers such that

$$(1,1,1) = \alpha(1,0,1) + \beta(0,1,2) + \gamma(-1,1,0).$$

Then

$$(1,1,1) = (\alpha - \gamma, \beta + \gamma, \alpha + 2\beta).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha \qquad\quad - \gamma &= 1 \\ \beta + \gamma &= 1 \\ \alpha + 2\beta \qquad &= 1. \end{aligned}$$

Adding the first two equations gives $\alpha + \beta = 2$, and solving this and the last equation gives $\beta = -1$ and $\alpha = 3$. Substitution then gives $\gamma = 2$, so the required linear combination is

$$(1,1,1) = 3(1,0,1) - 1(0,1,2) + 2(-1,1,0).$$

(You may have used Gauss–Jordan elimination to solve the system of linear equations, rather than solving them directly. Either method is fine.)

(b) Let α and β be real numbers such that

$$2 + 2x + 5x^2 = \alpha(1 + 3x^2) + \beta(2x - x^2).$$

Then

$$2 + 2x + 5x^2 = \alpha + (2\beta)x + (3\alpha - \beta)x^2.$$

Equating corresponding coefficients, we obtain the system

$$\begin{aligned} \alpha \qquad &= 2 \\ 2\beta &= 2 \\ 3\alpha - \quad \beta &= 5. \end{aligned}$$

The solutions can be read off from the first two equations, but it is important to check that *all* the equations are satisfied: otherwise there is no solution.

The first two equations have the solution $\alpha = 2$, $\beta = 1$, and this solution also satisfies the third equation. So the required linear combination is

$$2 + 2x + 5x^2 = 2(1 + 3x^2) + (2x - x^2).$$

Exercise C51

(a) In $\mathbb{R}^2$, express the vector $(2, 4)$ as a linear combination of the vectors $(0, 3)$ and $(2, 1)$.

(b) In $\mathbb{R}^3$, express the vector $(2, 3, -2)$ as a linear combination of the vectors $(0, 1, 0)$, $(1, 2, -1)$ and $(1, 1, -2)$.

(c) In $M_{2,2}$, express the matrix $\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$ as a linear combination of the matrices $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 0 & -2 \\ 0 & 1 \end{pmatrix}$.

2.2 Spanning sets

We now look at the set of vectors that is produced when we form *all possible* linear combinations of a given set of vectors.

Picture any two vectors in $\mathbb{R}^2$, and suppose that we form all possible linear combinations of these two vectors. What vectors do we obtain? Are there any vectors in $\mathbb{R}^2$ that *cannot* be written as a linear combination of these two vectors? (We saw such an example in Worked Exercise C27(b).) What happens if we start with one vector in $\mathbb{R}^2$? If we form all possible linear combinations of it, what vectors can result? What happens if we start with one, two or three vectors in $\mathbb{R}^3$?

Let us start with a set consisting of exactly one vector in $\mathbb{R}^2$ – namely, the set containing the vector $(1,0)$. The set of all linear combinations of $(1,0)$, illustrated in Figure 5, is

$$\{\alpha(1,0) : \alpha \in \mathbb{R}\} = \{(\alpha, 0) : \alpha \in \mathbb{R}\}.$$

Figure 5 The linear combinations of $(1,0)$

Geometrically, the members of this set are the points on the x-axis in $\mathbb{R}^2$. So this set of linear combinations is a line (the x-axis) in $\mathbb{R}^2$. We say that the set $\{(1,0)\}$ *spans* the x-axis, and that the x-axis is *spanned* by $\{(1,0)\}$.

Suppose that we now take the set $\{(1,0),(0,1)\}$ containing two vectors. The set of all linear combinations of $(1,0)$ and $(0,1)$, illustrated in Figure 6, is

$$\{\alpha(1,0) + \beta(0,1) : \alpha, \beta \in \mathbb{R}\} = \{(\alpha, \beta) : \alpha, \beta \in \mathbb{R}\}.$$

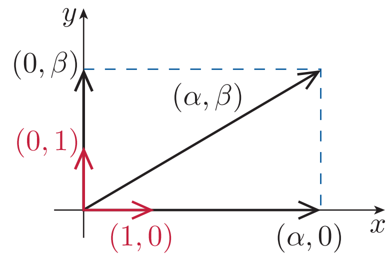


Figure 6 The linear combinations of $(1,0)$ and $(0,1)$

Since α and β can take any real values, this set consists of all the points in $\mathbb{R}^2$. We say that $\{(1,0),(0,1)\}$ *spans* $\mathbb{R}^2$, and that $\mathbb{R}^2$ is *spanned* by $\{(1,0),(0,1)\}$.

We now write down the formal definitions of span and spanning, before looking at some more examples.

Definitions

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a finite set of vectors in a vector space V. Then the **span** $\langle S \rangle$ of S is the set of all possible linear combinations

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k,$$

where $\alpha_1, \alpha_2, \dots, \alpha_k$ are real numbers; that is,

$$\langle S \rangle = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k : \alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}\}.$$

We say that the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ **spans** $\langle S \rangle$ or is a **spanning set** for $\langle S \rangle$, and that $\langle S \rangle$ is the set **spanned** by S.

While S is a *finite* set of vectors, the span $\langle S \rangle$ is generally an infinite set of vectors (such as a line or plane): this is because the linear combinations involve the set of real numbers. In fact, the span $\langle S \rangle$ is itself a vector space, as you will see later, in Subsection 4.1 (Theorem C28).

To test whether a vector $\mathbf{v}$ lies in the span of a given set S, we use Strategy C6 to determine whether $\mathbf{v}$ can be written as a linear combination of the vectors in S.

Worked Exercise C29

Let $S = \{(1,1,0), (0,1,1)\}$. Which of the following vectors belong to $\langle S\rangle$?

(a) $(0,0,1)$ (b) $(4,2,-2)$

Solution

We apply Strategy C6.

(a) We write

$$(0,0,1) = \alpha(1,1,0) + \beta(0,1,1) = (\alpha, \alpha+\beta, \beta).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha &= 0 \\ \alpha + \beta &= 0 \\ \beta &= 1. \end{aligned}$$

Subtracting the first and third equations from the second yields the equation $0 = -1$.

This system is inconsistent and therefore has no solution. So $(0,0,1)$ does not belong to $\langle S\rangle$.

(b) We write

$$(4,2,-2) = \alpha(1,1,0) + \beta(0,1,1) = (\alpha, \alpha+\beta, \beta).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha &= 4 \\ \alpha + \beta &= 2 \\ \beta &= -2. \end{aligned}$$

The first and third equations give $\alpha = 4$ and $\beta = -2$, and these values also satisfy the second equation. So $(4,2,-2)$ belongs to $\langle S\rangle$ and it can be written as

$$(4,2,-2) = 4(1,1,0) - 2(0,1,1).$$

Exercise C52

Let $\mathbf{v}_1 = (1,0,3)$, $\mathbf{v}_2 = (0,2,0)$ and $\mathbf{v}_3 = (0,3,1)$ be three vectors in $\mathbb{R}^3$. Use Strategy C6 to determine whether the vector $(1,5,4)$ lies in the subset of $\mathbb{R}^3$ spanned by each of the following sets.

(a) $\{\mathbf{v}_1, \mathbf{v}_2\}$ (b) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

Strategy C6 can also be used to show that a given set of vectors is a spanning set for the whole of a particular vector space, as we show in the following worked exercise.

Worked Exercise C30

Show that each of the following is a spanning set for $\mathbb{R}^2$.

(a) $\{(1,2),(2,-3)\}$ (b) $\{(1,0),(1,1),(1,-2)\}$

Solution

We need to show that *every* vector in $\mathbb{R}^2$ can be expressed as a linear combination of the given vectors, so we show that the general vector (x,y) can be.

(a) Each vector in $\mathbb{R}^2$ can be written as (x,y). To show that (x,y) is in $\langle\{(1,2),(2,-3)\}\rangle$, we write

$$(x,y)=\alpha(1,2)+\beta(2,-3)=(\alpha+2\beta,2\alpha-3\beta).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha+2\beta&=x\\ 2\alpha-3\beta&=y, \end{aligned}$$

whose solutions are $\alpha=\frac{1}{7}(3x+2y)$, $\beta=\frac{1}{7}(2x-y)$. So any vector in $\mathbb{R}^2$ can be written in terms of $(1,2)$ and $(2,-3)$ as

$$(x,y)=\tfrac{1}{7}(3x+2y)(1,2)+\tfrac{1}{7}(2x-y)(2,-3).$$

Thus $\{(1,2),(2,-3)\}$ is a spanning set for $\mathbb{R}^2$; that is,

$$\langle\{(1,2),(2,-3)\}\rangle=\mathbb{R}^2.$$

(b) Each vector in $\mathbb{R}^2$ can be written as (x,y). To show that (x,y) is in $\langle\{(1,0),(1,1),(1,-2)\}\rangle$, we write

$$\begin{aligned}(x,y)&=\alpha(1,0)+\beta(1,1)+\gamma(1,-2)\\ &=(\alpha+\beta+\gamma,\beta-2\gamma).\end{aligned}$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha+\beta+\ \ \gamma&=x\\ \beta-2\gamma&=y. \end{aligned}$$

We saw in Unit C1 that a consistent system of m equations in n unknowns, with $m<n$, has an infinite solution set.

This is a system of two linear equations in three unknowns, so if there is a solution, there will be infinitely many solutions.

We need just one solution, so try to simplify things by setting $\gamma=0$.

For example, taking $\gamma=0$ gives $\beta=y$ and $\alpha=x-y$. So

$$(x,y)=(x-y)(1,0)+y(1,1)+0(1,-2).$$

Thus $\langle\{(1,0),(1,1),(1,-2)\}\rangle=\mathbb{R}^2$.

The solution to Worked Exercise C30(b) shows that the set $\{(1,0),(1,1)\}$ is a spanning set for $\mathbb{R}^2$ so, in some sense, the vector $(1,-2)$ is redundant. We return to this idea of redundant vectors in a spanning set in the next section.

Exercise C53

Show that each of the following is a spanning set for $\mathbb{R}^2$.

(a) $\{(1,1),(-1,2)\}$ (b) $\{(2,-1),(3,2)\}$

Exercise C54

Show that $\{(1,0,0),(1,1,0),(2,0,1)\}$ is a spanning set for $\mathbb{R}^3$.

The following worked exercise shows that Strategy C6 can be used for vector spaces other than $\mathbb{R}^2$ and $\mathbb{R}^3$.

Worked Exercise C31

Show that $\{1+x^2, x^2, 2-x\}$ is a spanning set for P_3.

Solution

As before, we need to show that *every* polynomial in P_3 can be expressed as a linear combination of the given polynomials, so we show that the general polynomial $a+bx+cx^2$ can be.

Each polynomial in P_3 can be written as $a+bx+cx^2$. To show that $a+bx+cx^2$ is in $\langle\{1+x^2, x^2, 2-x\}\rangle$, we write

$$\begin{aligned} a+bx+cx^2 &= \alpha(1+x^2)+\beta(x^2)+\gamma(2-x) \\ &= \alpha+2\gamma-\gamma x+(\alpha+\beta)x^2. \end{aligned}$$

Equating corresponding coefficients, we obtain the system

$$\begin{aligned} \alpha \qquad + 2\gamma &= a \\ -\gamma &= b \\ \alpha+\beta \qquad &= c. \end{aligned}$$

It follows from the second equation that $\gamma=-b$. Substituting this into the first equation gives $\alpha=a+2b$ and hence, from the third equation, $\beta=c-a-2b$. So

$$a+bx+cx^2=(a+2b)(1+x^2)+(c-a-2b)x^2-b(2-x).$$

Thus $\langle\{1+x^2, x^2, 2-x\}\rangle = P_3$.

Exercise C55

Show that $\{1+x, 1+x^2, 1+x^3, x\}$ is a spanning set for P_4.

We look now at sets S in vector spaces V for which $\langle S\rangle$ is not the whole of V.

Worked Exercise C32

For each of the following vector spaces V and sets of vectors S in V, determine $\langle S\rangle$. In parts (a) and (b), describe $\langle S\rangle$ geometrically.

(a) $V = \mathbb{R}^2$, $\quad S = \{(1,1)\}$.

(b) $V = \mathbb{R}^3$, $\quad S = \{(1,0,1), (2,0,3)\}$.

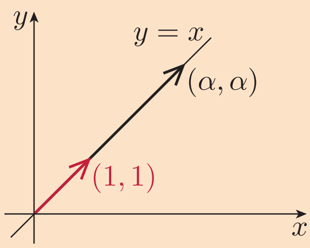
(c) $V = M_{2,3}$, $\quad S = \left\{ \begin{pmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \right\}$.

Solution

(a) We have

$$\langle S\rangle = \{\alpha(1,1) : \alpha \in \mathbb{R}\} = \{(\alpha,\alpha) : \alpha \in \mathbb{R}\}.$$

A picture can help.



Geometrically, $\langle S\rangle$ is the line $y = x$.

(b) We have

$$\begin{aligned}\langle S\rangle &= \{\alpha(1,0,1) + \beta(2,0,3) : \alpha,\beta \in \mathbb{R}\} \\ &= \{(\alpha+2\beta, 0, \alpha+3\beta) : \alpha,\beta \in \mathbb{R}\}.\end{aligned}$$

Every point in this set is of the form $(x,0,z)$.

Thus

$$\langle S\rangle \subseteq \{(x,0,z) : x,z \in \mathbb{R}\}.$$

To determine whether $\langle S\rangle$ is equal to this set we have to show that *every* vector $(x,0,z)$ can be expressed as a linear combination of $(1,0,1)$ and $(2,0,3)$.

To show that every vector $(x, 0, z)$, where $x, z \in \mathbb{R}$, belongs to $\langle S \rangle$, we write

$$(x, 0, z) = (\alpha + 2\beta, 0, \alpha + 3\beta).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha + 2\beta &= x \\ \alpha + 3\beta &= z. \end{aligned}$$

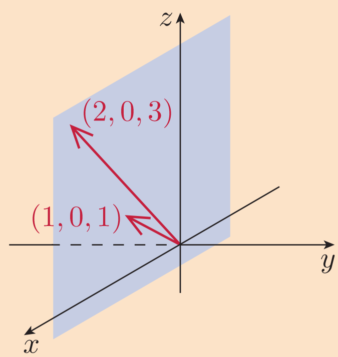
The solution is $\beta = z - x$ and $\alpha = 3x - 2z$, so

$$(x, 0, z) = (3x - 2z)(1, 0, 1) + (z - x)(2, 0, 3).$$

Hence $(x, 0, z) \in \langle S \rangle$, so any vector of the form $(x, 0, z)$ can be written in terms of $(1, 0, 1)$ and $(2, 0, 3)$. It follows that

$$\langle S \rangle = \{(x, 0, z) : x, z \in \mathbb{R}\}.$$

A picture can help.



Geometrically, $\langle S \rangle$ is the plane $y = 0$.

(c) We have

$$\begin{aligned} \langle S \rangle = \Bigg\{ &\alpha \begin{pmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \\ &+ \gamma \begin{pmatrix} 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \Bigg\} \\ = \Bigg\{ &\begin{pmatrix} 2\alpha + \beta & -\alpha - 2\gamma & 3\beta + 2\gamma \\ 0 & 0 & 0 \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \Bigg\}. \end{aligned}$$

Every matrix in this set is of the form $\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix}$.

Thus

$$\langle S \rangle \subseteq \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

To determine whether $\langle S\rangle$ is equal to this set we have to show that *every* matrix of this form can be expressed as a linear combination of the three given matrices.

To show that every 2×3 matrix with zero entries in the second row belongs to $\langle S\rangle$, we write

$$\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2\alpha+\beta & -\alpha-2\gamma & 3\beta+2\gamma \\ 0 & 0 & 0 \end{pmatrix}.$$

Equating corresponding entries, we obtain the system

$$\begin{aligned} 2\alpha + \beta \phantom{{}+2\gamma} &= a \\ -\alpha \phantom{{}+3\beta} - 2\gamma &= b \\ 3\beta + 2\gamma &= c. \end{aligned}$$

It has solution

$$\alpha = \tfrac{1}{7}(3a-b-c),$$
$$\beta = \tfrac{1}{7}(a+2b+2c),$$
$$\gamma = -\tfrac{1}{14}(3a+6b-c),$$

so

$$\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} \in \langle S\rangle.$$

Hence

$$\langle S\rangle = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} : a,b,c \in \mathbb{R} \right\}.$$

Exercise C56

For each of the following vector spaces V and sets of vectors S in V, determine $\langle S\rangle$.

(a) $V = \mathbb{R}^3, \quad S = \{(1,0,0)\}$.

(b) $V = M_{2,2}, \quad S = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \right\}$.

3 Bases and dimension

In this section you will see that there is a minimum number of vectors needed to span a vector space.

3.1 Linear independence and dependence

In Section 2 we found several spanning sets for $\mathbb{R}^2$ and $\mathbb{R}^3$. For example, in Worked Exercise C30(b), we showed that each of the sets

$$\{(1,0),(1,1)\} \quad \text{and} \quad \{(1,0),(1,1),(1,-2)\}$$

spans $\mathbb{R}^2$. In order to be able to work efficiently with a vector space, we need to express each vector in it as a linear combination of a small number of vectors. In particular, it would be convenient if we could find a set containing the *smallest* number of vectors that spans the space – that is, we want to find a *minimal spanning set*.

The set $\{(1,0),(1,1),(1,-2)\}$ is clearly not a minimal spanning set for $\mathbb{R}^2$, since the smaller set $\{(1,0),(1,1)\}$ also spans $\mathbb{R}^2$. The vector $(1,-2)$ is redundant because it can be written as a linear combination of the vectors $(1,0)$ and $(1,1)$:

$$(1,-2) = 3(1,0) - 2(1,1).$$

Thus, if a vector (x,y) in $\mathbb{R}^2$ can be written as a linear combination of the vectors $(1,0)$, $(1,1)$ and $(1,-2)$, then it can be written as a linear combination of just the vectors $(1,0)$ and $(1,1)$:

$$\begin{aligned}(x,y) &= \alpha(1,0) + \beta(1,1) + \gamma(1,-2)\\ &= \alpha(1,0) + \beta(1,1) + \gamma[3(1,0) - 2(1,1)]\\ &= (\alpha + 3\gamma)(1,0) + (\beta - 2\gamma)(1,1).\end{aligned}$$

The following general result holds.

Theorem C20

Suppose that the vector $\mathbf{v}_k$ can be written as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}$. Then the span of the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is the same as the span of the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}\}$.

Proof Let $S = \langle\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}\}\rangle$ and $T = \langle\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}\rangle$.

Clearly, $S \subseteq T$.

Now

$$T = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k : \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}\}.$$

As $\mathbf{v}_k$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}$, it follows that

$$\mathbf{v}_k = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_{k-1}\mathbf{v}_{k-1}, \text{ for some } \beta_1, \beta_2, \ldots, \beta_{k-1} \in \mathbb{R}.$$

So any vector of T can be expressed in the form

$$\begin{aligned}&\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\cdots+\alpha_k\mathbf{v}_k\\&\quad=\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\cdots+\alpha_{k-1}\mathbf{v}_{k-1}\\&\qquad+\alpha_k(\beta_1\mathbf{v}_1+\beta_2\mathbf{v}_2+\cdots+\beta_{k-1}\mathbf{v}_{k-1})\\&\quad=(\alpha_1+\alpha_k\beta_1)\mathbf{v}_1+(\alpha_2+\alpha_k\beta_2)\mathbf{v}_2+\cdots+(\alpha_{k-1}+\alpha_k\beta_{k-1})\mathbf{v}_{k-1},\end{aligned}$$

which belongs to S. Thus $T\subseteq S$.

Combining these two results gives $S=T$, as required. ■

So, in order to tell whether a spanning set is minimal, we need to be able to test whether *every* vector in the set can be written as a linear combination of the remaining vectors in the set. To make this task easier, we introduce the ideas of *linear dependence* and *linear independence*.

Definitions

A finite set of vectors $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$ in a vector space V is **linearly dependent** if there exist real numbers $\alpha_1,\alpha_2,\ldots,\alpha_k$, *not all zero*, such that

$$\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\cdots+\alpha_k\mathbf{v}_k=\mathbf{0}.$$

A finite set of vectors $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$ is **linearly independent** if it is not linearly dependent; that is, if

$$\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\cdots+\alpha_k\mathbf{v}_k=\mathbf{0}$$

only when $\alpha_1=\alpha_2=\cdots=\alpha_k=0$.

Note that $\alpha_1=\alpha_2=\cdots=\alpha_k=0$ *is* a solution to the equation whether the set of vectors is linearly dependent or linearly independent. So the distinction between the two cases is whether there is a *non-zero* solution.

We use the term *linearly dependent* because if a set of vectors is linearly dependent, then one of the vectors can be written as a linear combination of the others – that is, this vector *depends* on the others. If

$$\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\cdots+\alpha_k\mathbf{v}_k=\mathbf{0},$$

and α_k (for example) is non-zero, then we can rearrange the equation to give

$$\mathbf{v}_k=-\frac{\alpha_1}{\alpha_k}\mathbf{v}_1-\cdots-\frac{\alpha_{k-1}}{\alpha_k}\mathbf{v}_{k-1},$$

so that $\mathbf{v}_k$ is a linear combination of the remaining vectors. Hence $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$ is a linearly dependent set.

For example, if $2\mathbf{v}_1+3\mathbf{v}_2-4\mathbf{v}_3=\mathbf{0}$, then $\mathbf{v}_3=\frac{1}{2}\mathbf{v}_1+\frac{3}{4}\mathbf{v}_2$. In this case, $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is a linearly dependent set. We can also write $\mathbf{v}_1$ in terms of $\mathbf{v}_2$ and $\mathbf{v}_3$, and similarly $\mathbf{v}_2$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_3$.

Conversely, if one of a set of vectors can be written as a linear combination of the others, then the set is linearly dependent; that is, if $\mathbf{v}_k$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a linearly dependent set.

Statements 1 to 4 below follow from the definitions.

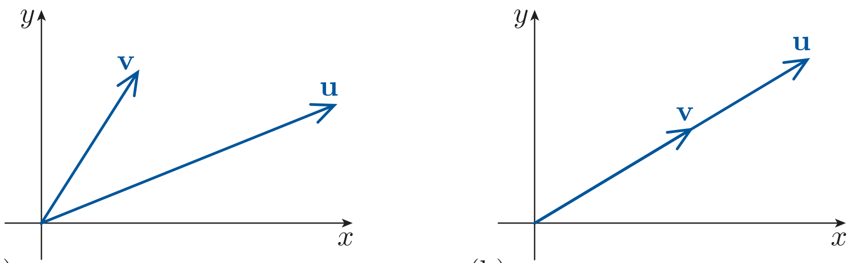
1. If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a linearly independent set, then there is only one way in which the zero vector can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$; that is, the trivial way
$$\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_k.$$
2. If $\mathbf{v}_1$ is the zero vector, then for $\alpha \in \mathbb{R}$,
$$\alpha\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_k = \mathbf{0},$$
so any set of vectors containing the zero vector is linearly dependent. It follows that *a linearly independent set cannot contain the zero vector*.
3. Any set consisting of just one non-zero vector $\mathbf{v}$ is linearly independent because if $\alpha\mathbf{v} = \mathbf{0}$, then either $\alpha = 0$ or $\mathbf{v} = \mathbf{0}$. Since $\mathbf{v}$ is non-zero, we must have $\alpha = 0$, so the set $\{\mathbf{v}\}$ is linearly independent.
4. Any set of two non-zero vectors is linearly dependent if one of the vectors is a multiple of the other, and linearly independent otherwise. This applies to vectors in all vector spaces: it is not restricted to vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

As an example of statement 4, consider the set $\{(1,1,2),(2,2,4)\}$ in $\mathbb{R}^3$. We have
$$(2,2,4) = 2(1,1,2),$$
so
$$-2(1,1,2) + (2,2,4) = (0,0,0),$$
which is the zero vector in $\mathbb{R}^3$. In this case $\alpha_1 = -2$ and $\alpha_2 = 1$. So this set is linearly dependent.

Similarly, $\{3 - 2x + x^2, 6 - 4x + 2x^2\}$ is a linearly dependent set in P_3 because
$$6 - 4x + 2x^2 = 2(3 - 2x + x^2),$$
so
$$2(3 - 2x + x^2) - (6 - 4x + 2x^2) = 0 + 0x + 0x^2,$$
which is the zero vector in P_3. In this case $\alpha_1 = 2$ and $\alpha_2 = -1$.

However, neither $\{(1,1,2),(1,2,-3)\}$ nor $\{3 - 2x + x^2, -1 + x + 2x^2\}$ is a linearly dependent set, as in each case neither vector is a multiple of the other.

Statement 4 therefore gives us a particularly simple way of checking whether a set of two non-zero vectors is linearly dependent or linearly independent: namely, a set of two non-zero vectors is linearly independent if and only if neither vector is a multiple of the other. For vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, this is equivalent to saying that two non-zero vectors are linearly independent if and only if they do not lie along the same straight line – that is, they are not *collinear*, as illustrated in Figure 7.

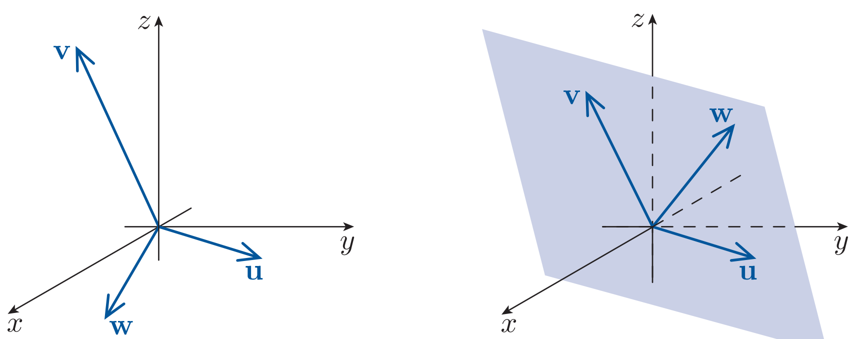


(a) (b)

Figure 7 Two vectors in $\mathbb{R}^2$ that are (a) linearly independent (b) linearly dependent

In this geometric interpretation of $\mathbb{R}^2$ a vector (x, y) is the *position vector* (x, y), not the point with coordinates (x, y). Therefore 'being collinear' is a property of the vectors (position vectors), not the points with these coordinates. For example, the two *points* $(1, 0)$ and $(1, 1)$ are collinear since they lie on the line $x = 1$, whereas the *vectors* $(1, 0)$ and $(1, 1)$ are not collinear since they are not multiples of one another and they do not both lie on a line through the origin: they are linearly independent vectors. By their definition as position vectors, collinear vectors will always lie on a line through the origin.

Similarly, three non-zero vectors in $\mathbb{R}^3$ are linearly independent if and only if they do not lie in the same plane – that is, they are not *coplanar*, as illustrated in Figure 8. In this geometric interpretation of $\mathbb{R}^3$ 'being coplanar' is again a property of the vectors (position vectors) not the points, so coplanar vectors in $\mathbb{R}^3$ will always lie on a plane through the origin.



(a) (b)

Figure 8 Three vectors in $\mathbb{R}^3$ that are (a) linearly independent (b) linearly dependent

More generally, we can use the following strategy to test whether a set of vectors is linearly independent.

Strategy C7

To test whether a given set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is linearly independent:

1. write down the equation $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k = \mathbf{0}$
2. express this equation as a system of linear equations in the unknowns $\alpha_1, \alpha_2, \dots, \alpha_k$
3. solve the resulting system of equations.

If the only solution is $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$, then the set of vectors is linearly independent.

If there is a solution with at least one of $\alpha_1, \alpha_2, \dots, \alpha_k$ not equal to zero, then the set of vectors is linearly dependent.

Worked Exercise C33

Use Strategy C7 to determine whether each of the following sets of vectors in $\mathbb{R}^3$ is linearly independent.

(a) $\{(2,0,0), (0,0,1), (-1,2,1)\}$ (b) $\{(1,1,1), (0,2,1), (1,5,3)\}$

Solution

We follow the steps of Strategy C7.

(a) We write $\alpha(2,0,0) + \beta(0,0,1) + \gamma(-1,2,1) = (0,0,0)$.

This simplifies to $(2\alpha - \gamma, 2\gamma, \beta + \gamma) = (0,0,0)$. Equating corresponding coordinates gives the equations we need.

This gives the system of linear equations

$$\begin{aligned} 2\alpha \qquad\; - \;\; \gamma &= 0 \\ 2\gamma &= 0 \\ \beta + \;\; \gamma &= 0. \end{aligned}$$

The second equation gives $\gamma = 0$. Substituting this value into the other two equations gives $\alpha = 0$ and $\beta = 0$. The only solution is $\alpha = \beta = \gamma = 0$.

Therefore this set of vectors is linearly independent.

(b) We write $\alpha(1,1,1) + \beta(0,2,1) + \gamma(1,5,3) = (0,0,0)$.

This gives the system of linear equations

$$\begin{aligned} \alpha \qquad\; + \;\; \gamma &= 0 \\ \alpha + 2\beta + 5\gamma &= 0 \\ \alpha + \;\; \beta + 3\gamma &= 0. \end{aligned}$$

A solution is not so easy to see, so we use the method of Gauss–Jordan elimination from Unit C1.

We perform row-reduction on the augmented matrix for this system of linear equations.

$$\begin{array}{ll} \begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} & \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right) \begin{array}{c} 2 \\ 8 \\ 5 \end{array} \\[2ex] \begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1 \end{array} & \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \begin{array}{c} 2 \\ 6 \\ 3 \end{array} \\[2ex] \begin{array}{l} \\ \mathbf{r}_2 \to \frac{1}{2}\mathbf{r}_2 \\ \\ \end{array} & \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \begin{array}{c} 2 \\ 3 \\ 3 \end{array} \\[2ex] \begin{array}{l} \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_2 \end{array} & \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} 2 \\ 3 \\ 0 \end{array} \end{array}$$

The corresponding system of equations is

$$\begin{aligned} \alpha \qquad + \ \ \gamma &= 0 \\ \beta + 2\gamma &= 0. \end{aligned}$$

The solution set of the system is

$$\alpha = -k, \ \beta = -2k, \ \gamma = k, \quad k \in \mathbb{R},$$

so there are infinitely many solutions. For example, $k = -1$ gives

$$(1,1,1) + 2(0,2,1) - (1,5,3) = (0,0,0).$$

So this set of vectors is linearly dependent.

Any one of the vectors can be written as a linear combination of the other two, for example $(1,1,1) = (1,5,3) - 2(0,2,1)$.

We claimed earlier that three non-zero linearly dependent vectors in $\mathbb{R}^3$ are coplanar and this was the case in Worked Exercise C33(b). You may like to check that all the vectors in the set lie in the plane through the origin with equation $x + y - 2z = 0$.

In the following exercise you are asked to determine whether given sets of vectors are linearly independent or not. Before embarking on the algebra, have a look at each set of vectors and try to decide whether you expect the set to be linearly dependent or linearly independent; it may be that Strategy C7 is not needed in some cases.

Exercise C57

Determine whether each of the following sets of vectors is a linearly independent subset of V.

(a) $V = \mathbb{R}^2, \quad \{(1,0),(-1,-1)\}$.

(b) $V = \mathbb{R}^2, \quad \{(1,-1),(1,1),(2,1)\}$.

(c) $V = \mathbb{R}^3, \quad \{(1,1,0),(-1,1,1)\}$.

(d) $V = \mathbb{R}^3, \quad \{(1,0,0),(1,1,0),(1,1,1)\}$.

(e) $V = \mathbb{R}^4, \quad \{(1,2,1,0),(0,-1,1,3)\}$.

We conclude this subsection by looking briefly at linearly dependent and linearly independent sets of vectors in vector spaces other than $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^4$. Again, before embarking on the algebra, it is sensible to have a look at each set of vectors: it may be that Strategy C7 is not needed in some cases.

Worked Exercise C34

Determine whether the set of polynomials $\{1, 4x, 4x + x^2\}$ is a linearly independent subset of P_3.

Solution

There is no obvious linear dependence.

We apply Strategy C7.

We write $\alpha(1) + \beta(4x) + \gamma(4x + x^2) = 0$, which can be written as

$$\alpha + (4\beta + 4\gamma)x + \gamma x^2 = 0 + 0x + 0x^2.$$

Equating coefficients, we obtain the system

$$\begin{aligned} \alpha \qquad\qquad &= 0 \\ 4\beta + 4\gamma &= 0 \\ \gamma &= 0. \end{aligned}$$

So $\alpha = 0$, $\gamma = 0$ and, by substitution in the second equation, $\beta = 0$. The only soluton is $\alpha = \beta = \gamma = 0$.

Therefore $\{1, 4x, 4x + x^2\}$ is a linearly independent subset of P_3.

Worked Exercise C35

In each case, determine whether the set S of matrices is a linearly independent subset of $M_{2,2}$.

(a) $S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix} \right\}$

(b) $S = \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -2 & 2 \\ 0 & -4 \end{pmatrix} \right\}$

(c) $S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \right\}$

Solution

(a) There are just two matrices and neither is a multiple of the other, so the strategy is unnecessary.

The set S is linearly independent because neither matrix is a multiple of the other.

(b) The second matrix is a multiple of the first (-2 times), so the strategy is unnecessary.

The set S is linearly dependent because

$$2\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -2 & 2 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(c) There is no obvious linear dependence.

We apply Strategy C7.

We write

$$\alpha\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} + \beta\begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix} + \gamma\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

which can be written as

$$\begin{pmatrix} \alpha + 2\gamma & \alpha - \beta + 3\gamma \\ -2\beta + 2\gamma & 2\alpha + \beta + 3\gamma \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Equating corresponding entries, we obtain the system

$$\begin{aligned} \alpha \qquad\quad + 2\gamma &= 0 \\ \alpha - \quad \beta + 3\gamma &= 0 \\ - 2\beta + 2\gamma &= 0 \\ 2\alpha + \quad \beta + 3\gamma &= 0. \end{aligned}$$

The first and third equations both simply relate *two* unknowns, so it is sensible to start with these.

From the third equation we have $2\beta = 2\gamma$, that is, $\beta = \gamma$, and from the first equation $\alpha = -2\gamma$. If we choose $\gamma = 1$, then $\beta = 1$ and $\alpha = -2$, and these also satisfy the second and fourth equations; thus

$$-2\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} + 1\begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix} + 1\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So we can find α, β and γ not all zero such that the original equation is satisfied. So the set of matrices is linearly dependent. It is not a linearly independent subset of $M_{2,2}$.

Exercise C58

In each of the following cases, determine whether S is a linearly independent subset of the vector space V.

(a) $V = P_4, \quad S = \{1, x, x^2, x^3, 1 + x + x^2 + x^3\}$.

(b) $V = M_{2,2}, \quad S = \left\{ \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \right\}$.

(c) $V = M_{2,2}, \quad S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$.

(d) $V = \mathbb{C}, \quad S = \{1 + i, 1 - i\}$.

3.2 Bases

We now use the idea of linear independence to help us find a minimal set of vectors that spans a vector space.

If we have a set of vectors that forms a spanning set for a vector space, then the set is a minimal spanning set if and only if *it is linearly independent*.

This condition is certainly necessary because, as we showed in the previous subsection, if the set of vectors is linearly dependent, then we can write at least one of the vectors as a linear combination of the other vectors. Such a vector is redundant, and we can drop it from the set, so the set is not a minimal set.

The condition is also sufficient; we prove this using proof by contradiction. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a linearly independent spanning set for a vector space V, and suppose that the smaller set $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}\}$ also spans V. This means we can write any vector in V as a linear combination of the vectors in S_1. In particular we can write

$$\mathbf{v}_k = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_{k-1} \mathbf{v}_{k-1},$$

for some $\alpha_1, \ldots, \alpha_{k-1}$ not all equal to 0. Therefore

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_{k-1} \mathbf{v}_{k-1} - \mathbf{v}_k = \mathbf{0},$$

so S is not linearly independent. But this is a contradiction, so our initial assumption that S_1 spans V must be wrong. Thus S_1 cannot span V and S is a minimal spanning set.

If we have a linearly independent set of vectors that spans a vector space, then we give the set of vectors a special name.

Definition

A **basis** for a vector space V is a linearly independent set of vectors that is a spanning set for V.

The plural of basis is **bases**. A basis of a vector space V is *one* set of linearly independent vectors that spans V; a basis is not unique, so V can have many different bases.

You saw in Exercise C53(a) that $\{(1,1),(-1,2)\}$ is a spanning set for $\mathbb{R}^2$. Since it is also a linearly independent set, it is a *basis* for $\mathbb{R}^2$. Although the set $\{(1,0),(1,1),(1,-2)\}$ is also a spanning set for $\mathbb{R}^2$, it is not linearly independent, as we showed earlier in this section: so it is not a basis for $\mathbb{R}^2$.

While each vector in $\mathbb{R}^2$ can be written as a linear combination of vectors in the spanning set $\{(1,0),(1,1),(1,-2)\}$, this expression is not unique. For example,

$$\begin{aligned}(0,1) &= 2(1,0) - 1(1,1) - 1(1,-2)\\ &= -4(1,0) + 3(1,1) + 1(1,-2).\end{aligned}$$

An important property of a basis for a vector space V is that each vector in V has a *unique* expression as a linear combination of basis vectors.

Theorem C21

Let S be a basis for a vector space V. Then each vector in V can be expressed as a linear combination of the vectors in S in only one way.

Proof Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for a vector space V.

We assume that a vector in V can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in two different ways, and show that this leads to a contradiction.

Let $\mathbf{u}$ be a vector in V, and assume that we can write $\mathbf{u}$ as a linear combination of the vectors in S in two different ways as:

$$\mathbf{u} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k$$

and

$$\mathbf{u} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_k\mathbf{v}_k.$$

Then

$$\mathbf{0} = \mathbf{u} - \mathbf{u} = (\alpha_1 - \beta_1)\mathbf{v}_1 + (\alpha_2 - \beta_2)\mathbf{v}_2 + \cdots + (\alpha_k - \beta_k)\mathbf{v}_k,$$

and $(\alpha_1 - \beta_1), (\alpha_2 - \beta_2), \ldots, (\alpha_k - \beta_k)$ are not all zero.

Therefore the set S is linearly dependent. But S is a basis for V, and is therefore linearly independent. This contradiction shows that Theorem C21 is true. ■

The definition of a basis gives us a strategy for testing whether a given set of vectors is a basis for a particular vector space.

Strategy C8

To determine whether a set of vectors S in a vector space V is a basis for V, check the following conditions.

(1) S is linearly independent.

(2) S spans V.

If both (1) and (2) hold, then S is a basis for V.

If either (1) or (2) does not hold, then S is not a basis for V.

Worked Exercise C36

Show that $S = \{(2,0,2),(1,1,1),(0,1,-1)\}$ is a basis for $\mathbb{R}^3$.

Solution

We check both conditions in Strategy C8.

We start by checking condition (1): S is linearly independent.

Using Strategy C7, we write

$$\alpha(2,0,2) + \beta(1,1,1) + \gamma(0,1,-1) = (0,0,0),$$

which simplifies to

$$(2\alpha+\beta, \beta+\gamma, 2\alpha+\beta-\gamma) = (0,0,0).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} 2\alpha + \beta \phantom{{}+\gamma} &= 0 \\ \beta + \gamma &= 0 \\ 2\alpha + \beta - \gamma &= 0. \end{aligned}$$

We could use Gauss–Jordan elimination, but we can solve this system directly.

Subtracting the third equation from the first gives $\gamma = 0$, and substituting this into the second equation gives $\beta = 0$. Finally, substituting $\beta = 0$ into the first equation gives $\alpha = 0$. The only solution is $\alpha = \beta = \gamma = 0$.

Therefore the set S is linearly independent.

We now check condition (2): S spans $\mathbb{R}^3$.

We apply Strategy C6.

We need to show that *every* vector in $\mathbb{R}^3$ can be expressed as a linear combination of the vectors in S, so we show that the general vector (x,y,z) can be.

Each vector in $\mathbb{R}^3$ can be written as (x, y, z), with $x, y, z \in \mathbb{R}$. To show that (x, y, z) is in $\langle S \rangle$, we write

$$(x, y, z) = \alpha(2, 0, 2) + \beta(1, 1, 1) + \gamma(0, 1, -1).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} 2\alpha + \beta \phantom{{}-\gamma} &= x \\ \beta + \gamma &= y \\ 2\alpha + \beta - \gamma &= z. \end{aligned}$$

Subtracting the third equation from the first gives $\gamma = x - z$, and substituting this into the second equation gives $\beta = y - x + z$. Finally, substituting for β in the first equation gives $\alpha = \frac{1}{2}(2x - y - z)$. We have a solution, so any vector in $\mathbb{R}^3$ can be written in terms of vectors in S as

$$\begin{aligned} (x, y, z) = {}&\tfrac{1}{2}(2x - y - z)(2, 0, 2) + (y - x + z)(1, 1, 1) \\ &+ (x - z)(0, 1, -1). \end{aligned}$$

Therefore S spans $\mathbb{R}^3$.

Since conditions (1) and (2) hold, the set S is a basis for $\mathbb{R}^3$.

Worked Exercise C37

Determine whether each of the following sets is a basis for $\mathbb{R}^3$.

(a) $\{(0, 1, 2), (1, 2, -1)\}$ (b) $\{(1, 1, 1), (0, 2, 1), (-1, 1, 0)\}$

Solution

(a) We check both conditions in Strategy C8.

The set $\{(0, 1, 2), (1, 2, -1)\}$ is linearly independent, as neither vector is a multiple of the other.

We apply Strategy C6.

We need to show that *every* vector in $\mathbb{R}^3$ can be expressed as a linear combination of the given vectors, so we show that the general vector can be.

Each vector in $\mathbb{R}^3$ can be written as (x, y, z), with $x, y, z \in \mathbb{R}$. To show that (x, y, z) is in $\langle\{(0, 1, 2), (1, 2, -1)\}\rangle$, we write

$$(x, y, z) = \alpha(0, 1, 2) + \beta(1, 2, -1).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \beta &= x \\ \alpha + 2\beta &= y \\ 2\alpha - \beta &= z. \end{aligned}$$

Substituting $\beta = x$ from the first equation into the other two equations gives

$$\alpha = y - 2x$$
$$\alpha = \tfrac{1}{2}(x + z).$$

The vector (x, y, z) is a general vector, so we need a solution for every possible combination of x, y and z.

These two equations are true simultaneously if and only if $y - 2x = \frac{1}{2}(x+z)$; that is, if and only if $5x - 2y + z = 0$.

This is not true for every x, y and z. In fact, it shows that $\langle\{(0,1,2),(1,2,-1)\}\rangle$ is the plane $5x - 2y + z = 0$ in $\mathbb{R}^3$; thus any point not on this plane cannot be written as a linear combination of the vectors $(0,1,2)$ and $(1,2,-1)$.

This contradicts the assumption that x, y and z can take any real values, so $\{(0,1,2),(1,2,-1)\}$ is not a spanning set for $\mathbb{R}^3$. Thus it is not a basis for $\mathbb{R}^3$.

(b) We check both conditions in Strategy C8.

Before diving into Strategy C7, we quickly look at the given vectors to see if there is any obvious linear dependence.

Here we have

$$(-1,1,0) = -(1,1,1) + (0,2,1),$$

so these vectors are not linearly independent.

Therefore the set $\{(1,1,1),(0,2,1),(-1,1,0)\}$ is not a basis for $\mathbb{R}^3$.

Exercise C59

Determine whether each of the following sets is a basis for $\mathbb{R}^3$.

(a) $\{(0,1,2),(0,2,3),(0,6,1)\}$

(b) $\{(1,2,1),(1,0,-1),(0,3,1)\}$

(c) $\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$

Exercise C60

Determine whether $\{(1,2,-1,-1),(-1,5,1,3)\}$ is a basis for $\mathbb{R}^4$.

We now consider bases for vector spaces other than $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^4$.

Worked Exercise C38

Determine whether each of the following sets is a basis for P_3.

(a) $\{1, x, x^2\}$ (b) $\{1, x\}$ (c) $\{1, 2 + x^2, x^2\}$

Solution

(a) We check both conditions in Strategy C8.

We check whether $\{1, x, x^2\}$ is linearly independent.

Using Strategy C7, we write

$$\alpha 1 + \beta x + \gamma x^2 = 0 + 0x + 0x^2.$$

Comparing coefficients, we have $\alpha = \beta = \gamma = 0$ as the only solution, so the set is linearly independent.

We check whether $\{1, x, x^2\}$ spans P_3.

We apply Strategy C6.

We need to show that *every* vector (polynomial) in P_3 can be written as a linear combination of 1, x and x^2, so we show that the general vector $a + bx + cx^2$ can be.

Each vector in P_3 can be written as $a + bx + cx^2$, with $a, b, c \in \mathbb{R}$. To show that $a + bx + cx^2$ is in $\langle\{1, x, x^2\}\rangle$, we write

$$a + bx + cx^2 = \alpha(1) + \beta(x) + \gamma(x^2).$$

Equating coefficients, we see that $a = \alpha$, $b = \beta$ and $c = \gamma$.

Therefore the set of vectors spans P_3.

Thus $\{1, x, x^2\}$ is a basis for P_3.

(b) Notice that x^2 cannot be expressed as a linear combination of 1 and x.

None of the vectors contains an x^2 term, so the set $\{1, x\}$ does not span P_3.

Therefore this set of vectors is not a basis for P_3.

You may have noticed that neither vector is a multiple of the other, so the set $\{1, x\}$ is linearly independent. The span of this set consists of polynomials of the form $a + bx$, which is a proper subset of P_3.

(c) Here we have

$$2 + x^2 = 2(1) + 1(x^2),$$

so the set $\{1, 2 + x^2, x^2\}$ is not linearly independent.

Therefore $\{1, 2 + x^2, x^2\}$ is not a basis for P_3.

The span of this set consists of all polynomials of the form $a + bx^2$, which again is a proper subset of P_3.

Exercise C61

Determine whether

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

is a basis for $M_{2,2}$.

3.3 Standard bases

You may have noticed that some sets of basis vectors seem to make the calculations in vector spaces particularly simple. For $\mathbb{R}^2$ this set is $\{(1,0),(0,1)\}$, for $\mathbb{R}^3$ it is $\{(1,0,0),(0,1,0),(0,0,1)\}$, and so on.

The representation of a vector in terms of these bases is straightforward. For example, in $\mathbb{R}^2$

$$(x,y) = x(1,0) + y(0,1),$$

and in $\mathbb{R}^3$

$$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1).$$

Because these bases are so simple, they are used frequently; they are called *standard bases*.

Definition

The **standard basis** for $\mathbb{R}^n$ is the set of n vectors

$$\{(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\}.$$

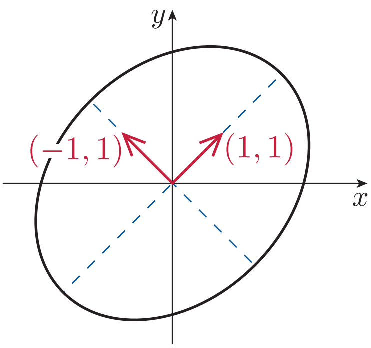


Figure 9 An ellipse with non-standard basis shown

The standard basis for $\mathbb{R}^n$ seems so natural that you may wonder why we do not use it all the time. In some physical situations, however, we may need to choose a different basis. For example, if we are looking at an ellipse centred at the origin, we may want to choose basis vectors along the major and minor axes of the ellipse. For the ellipse shown in Figure 9, it may be more convenient to choose the basis vectors $(1,1)$ and $(-1,1)$ rather than the standard ones, $(1,0)$ and $(0,1)$. Similarly, if we are considering a parallelogram, we may want to choose basis vectors along the sides of the parallelogram. In many vector spaces other than $\mathbb{R}^n$ there are particularly simple bases, which we call the standard bases for these spaces. Here are some examples.

$$P_n: \quad \{1, x, x^2, \ldots, x^{n-1}\}$$

$$M_{2,2}: \quad \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$\mathbb{C}: \quad \{1, i\}$$

If we write a vector in $\mathbb{R}^2$ as (x, y), then x and y are the components, or *coordinates*, of the vector with respect to the standard basis vectors – that is,

$$(x, y) = x(1, 0) + y(0, 1).$$

However, we need some way of indicating what the *coordinates* of a vector are with respect to non-standard basis vectors. We use the following notation.

Definitions

Let $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be a basis for a vector space V, and suppose that

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n,$$

where $v_1, v_2, \ldots, v_n \in \mathbb{R}$.

Then the ***E*-coordinate representation** of $\mathbf{v}$ is

$$\mathbf{v}_E = (v_1, v_2, \ldots, v_n)_E.$$

We call $v_1, v_2, \ldots, v_n$ the **coordinates of v with respect to the basis *E***, or, more briefly, the ***E*-coordinates** of $\mathbf{v}$.

Remarks

1. We usually omit the subscript if E is the standard basis.
2. We write the basis vectors as $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ rather than $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ to avoid confusion between the basis vectors and the coordinates $v_1, v_2, \ldots, v_n$ of a vector $\mathbf{v}$.
3. We can denote the E-coordinates of a vector $\mathbf{v}_j$ by $v_{1j}, v_{2j}, \ldots, v_{nj}$. So we write $\mathbf{v}_j = v_{1j}\mathbf{e}_1 + v_{2j}\mathbf{e}_2 + \cdots + v_{nj}\mathbf{e}_n$.
4. Since E is a basis for V, the E-coordinate representation of a vector in V is unique. However, the order of the coordinates in such a representation depends on the order of the basis vectors.
5. A non-zero vector has a different coordinate representation for each different basis. For the zero vector, the coordinates are always zero.

   You can think of the different representations of a vector as analogous to an amount of money being expressed in different currencies; in every currency, 'no money' is the same as 'zero money'.
6. If E is a standard basis, then we refer to the *standard coordinate representation*, *standard coordinates*, and so on.

The following worked exercise shows this notation in practice.

Worked Exercise C39

Given the basis $E = \{(-1,2),(2,2)\}$ for $\mathbb{R}^2$, determine the standard coordinate representation of $(3,2)_E$.

Solution

The coordinates $(3,2)_E$ are with respect to the basis E, meaning three times the first basis vector in E and twice the second.

For the basis $E = \{(-1,2),(2,2)\}$, we have

$$\begin{aligned}(3,2)_E &= 3(-1,2) + 2(2,2) \\ &= (-3,6) + (4,4) \\ &= (1,10).\end{aligned}$$

There is no subscript on the coordinates $(1,10)$ because they are with respect to the standard basis for $\mathbb{R}^2$.

Exercise C62

(a) Given the basis $E = \{(1,2),(-3,1)\}$ for $\mathbb{R}^2$, determine the standard coordinate representation of $(2,1)_E$.

(b) Given the basis $E = \{(1,0,2),(-1,1,3),(2,-2,0)\}$ for $\mathbb{R}^3$, determine the standard coordinate representation of $(1,1,-1)_E$.

We can also turn around the method in Worked Exercise C39 to express a given vector in terms of a non-standard basis.

Worked Exercise C40

For each of the following bases E for $\mathbb{R}^2$, find the E-coordinate representation of the vector $(1,4)$.

(a) $E = \{(1,4),(4,-1)\}$ (b) $E = \{(-1,2),(2,2)\}$

Solution

(a) We write $(1,4) = \alpha(1,4) + \beta(4,-1)$, which has the solution $\alpha = 1$, $\beta = 0$, so

$$(1,4) = 1(1,4) + 0(4,-1) = (1,0)_E.$$

(b) We write $(1,4) = \alpha(-1,2) + \beta(2,2) = (-\alpha + 2\beta, 2\alpha + 2\beta)$.

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} -\alpha + 2\beta &= 1 \\ 2\alpha + 2\beta &= 4. \end{aligned}$$

Solving these equations gives $\alpha = 1$ and $\beta = 1$, so

$$(1,4) = 1(-1,2) + 1(2,2) = (1,1)_E.$$

Geometrically, by changing the basis we are changing the axes we are using. For example, in Worked Exercise C40(b) we are expressing the vector $(1,4)$ (with respect to the standard basis) as a vector in terms of the new basis vectors $E = \{(-1,2),(2,2)\}$. The E-coordinates of this vector with respect to the basis E are $(1,1)_E$ representing one step along the $(-1,2)$-axis then one step along the $(2,2)$-axis. Figure 10 illustrates how this vector is represented with respect to these new axes.

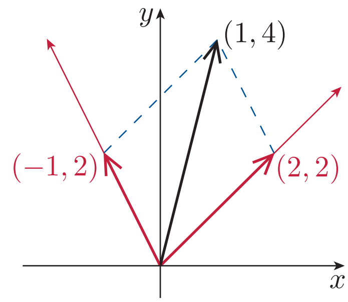


Figure 10 Changing the axes

Worked Exercise C41

Find the E-coordinate representation of the vector $(-2,0,1)$ with respect to the basis $E = \{(1,0,0),(1,0,1),(2,1,-1)\}$ for $\mathbb{R}^3$.

Solution

We write

$$\begin{aligned} (-2,0,1) &= \alpha(1,0,0) + \beta(1,0,1) + \gamma(2,1,-1) \\ &= (\alpha + \beta + 2\gamma, \gamma, \beta - \gamma). \end{aligned}$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha + \beta + 2\gamma &= -2 \\ \gamma &= 0 \\ \beta - \gamma &= 1. \end{aligned}$$

The second equation gives $\gamma = 0$. Substituting this value into the third equation gives $\beta = 1$, and substituting these values into the first equation gives $\alpha = -3$. So

$$\begin{aligned} (-2,0,1) &= -3(1,0,0) + 1(1,0,1) + 0(2,1,-1) \\ &= (-3,1,0)_E. \end{aligned}$$

Exercise C63

(a) Find the E-coordinate representation of the vector $(5, -4)$ with respect to the basis $E = \{(1, 2), (-3, 1)\}$ for $\mathbb{R}^2$.

(b) Find the E-coordinate representation of the vector $(-3, 5, 7)$ with respect to the basis $E = \{(1, 0, 2), (-1, 1, 3), (2, -2, 0)\}$ for $\mathbb{R}^3$.

3.4 Dimension

You may have noticed in the previous subsection that all the bases you met for $\mathbb{R}^2$ contained two vectors, all the bases for $\mathbb{R}^3$ contained three vectors, and so on. This should correspond to your intuitive idea of dimension – namely that $\mathbb{R}$ is one-dimensional, $\mathbb{R}^2$ is two-dimensional, and so on.

For example, among the bases you met were the following.

$$\mathbb{R}^2 : \{(1, 0), (0, 1)\}, \quad \{(1, 0), (1, 1)\}, \quad \{(1, 2), (-1, 1)\}.$$

$$\mathbb{R}^3 : \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}, \quad \{(1, 2, 1), (1, 0, -1), (0, 3, 1)\}.$$

$$\begin{aligned}\mathbb{R}^4 : &\{(1, 0, 2, 0), (0, 1, 0, 3), (0, 0, 1, 2), (2, 0, -1, 0)\},\\ &\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}.\end{aligned}$$

It is not a coincidence that every basis for $\mathbb{R}^2$ contains exactly two vectors, and every basis for $\mathbb{R}^3$ contains exactly three vectors. The main theorem in this section, the *Basis Theorem*, states that if V is any vector space, then *every basis for V contains the same number of vectors.* Before we prove this, we must define what we mean by a finite-dimensional vector space.

Definitions

Let V be a vector space. Then V is **finite-dimensional** if it contains a finite set of vectors S that forms a basis for V. If no such set exists, then V is **infinite-dimensional**.

Examples of infinite-dimensional vector spaces are $\mathbb{R}^\infty$ and the set of polynomials of any degree. On the other hand, the set containing just the zero vector is a zero-dimensional vector space, which has the empty set as its basis.

In order to prove that every basis for a finite-dimensional vector space V contains the same number of vectors, we first prove the following useful result.

Theorem C22

Let $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be a basis for a vector space V, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be a set of m vectors in V, where $m > n$. Then S is a linearly dependent set.

Proof We assume that the conditions of Theorem C22 hold and show that this implies that S is linearly dependent.

Let $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be a basis for V, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be a set of m vectors in V. Then each of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ can be written as a linear combination of the vectors in E; that is,

$$\begin{aligned}
\mathbf{v}_1 &= v_{11}\mathbf{e}_1 + v_{21}\mathbf{e}_2 + \cdots + v_{n1}\mathbf{e}_n,\\
\mathbf{v}_2 &= v_{12}\mathbf{e}_1 + v_{22}\mathbf{e}_2 + \cdots + v_{n2}\mathbf{e}_n,\\
&\ \vdots\\
\mathbf{v}_m &= v_{1m}\mathbf{e}_1 + v_{2m}\mathbf{e}_2 + \cdots + v_{nm}\mathbf{e}_n,
\end{aligned}$$

for some numbers $v_{11}, \ldots, v_{nm} \in \mathbb{R}$.

To show that S is linearly dependent, we must find real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$, not all zero, such that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_m\mathbf{v}_m = \mathbf{0}. \tag{1}$$

Using the first system of equations, we can rewrite equation (1) as

$$\begin{aligned}
&(\alpha_1 v_{11} + \alpha_2 v_{12} + \cdots + \alpha_m v_{1m})\mathbf{e}_1\\
&+ (\alpha_1 v_{21} + \alpha_2 v_{22} + \cdots + \alpha_m v_{2m})\mathbf{e}_2\\
&+ \cdots + (\alpha_1 v_{n1} + \alpha_2 v_{n2} + \cdots + \alpha_m v_{nm})\mathbf{e}_n = \mathbf{0}.
\end{aligned} \tag{2}$$

Since E is a basis, the set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is linearly independent. It follows that we can find real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$, not all zero, that satisfy equation (2) if and only if the following system of equations has a non-zero solution for $\alpha_1, \alpha_2, \ldots, \alpha_m$:

$$\begin{aligned}
v_{11}\alpha_1 + v_{12}\alpha_2 + \cdots + v_{1m}\alpha_m &= 0\\
v_{21}\alpha_1 + v_{22}\alpha_2 + \cdots + v_{2m}\alpha_m &= 0\\
&\ \vdots\\
v_{n1}\alpha_1 + v_{n2}\alpha_2 + \cdots + v_{nm}\alpha_m &= 0.
\end{aligned}$$

This is a system of n linear equations in m unknowns with $m > n$, so there are more unknowns than equations.

In Unit C1 you saw that a consistent system with more unknowns than equations has an infinite solution set. The system above is consistent because it is homogeneous, and therefore it has an infinite solution set.

Such a system of linear equations has a non-trivial solution – that is, a solution for which some variables are non-zero. Therefore the set S containing $m > n$ vectors is linearly dependent. This proves the theorem. ■

For example, $\mathbb{R}^3$ has three vectors in its standard basis, so, by Theorem C22, the set

$$\{(1,1,0),(0,-2,1),(0,0,1),(1,1,2)\}$$

is linearly dependent because it contains more than three vectors. In fact,

$$(1,1,0)+0(0,-2,1)+2(0,0,1)-(1,1,2)=(0,0,0).$$

Theorem C22 has the following immediate, and useful, consequence.

Corollary C23

Let V be a vector space with a basis containing n vectors. If a linearly independent subset of V contains m vectors, then $m \leq n$.

This corollary provides the crucial steps in the proof of the Basis Theorem.

Theorem C24 Basis Theorem

Let V be a finite-dimensional vector space. Then every basis for V contains the same number of vectors.

Proof We assume there are two bases with n and m vectors, respectively, and show that since a basis is a linearly independent set, this implies that $n = m$.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ be two bases for a finite-dimensional vector space V.

Since $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for V and $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ is a linearly independent set, we have $m \leq n$, by Corollary C23.

Similarly, since $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ is a basis for V and $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is linearly independent, we have $n \leq m$, by Corollary C23.

Therefore $m = n$, so every basis contains the same number of vectors. ■

The Basis Theorem allows us to give a definition of the dimension of a finite-dimensional vector space, which agrees with our intuitive idea of dimension.

Definition

The **dimension** of a finite-dimensional vector space V, denoted by $\dim V$, is the number of vectors in any basis for the space.

So $\mathbb{R}^2$ has dimension 2 and $\mathbb{R}^3$ has dimension 3, as we would expect. More generally, $\mathbb{R}^n$ has dimension n, since the standard basis for $\mathbb{R}^n$ has n vectors. It follows from Theorem C24 that every basis for $\mathbb{R}^n$ contains exactly n vectors. The strategy for checking whether a set of vectors is a basis (Strategy C8) can now be greatly simplified when the vector space is $\mathbb{R}^n$. The result that we need is stated in the next theorem.

Theorem C25

Let V be an n-dimensional vector space. Then any set of n linearly independent vectors in V is a basis for V.

Proof We give a proof by contradiction.

Suppose that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of n linearly independent vectors does not span V. Then there exists a vector $\mathbf{v}$ in V that cannot be written as a linear combination of the vectors in S.

So, if

$$\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n + \alpha_{n+1}\mathbf{v} = \mathbf{0},$$

then $\alpha_{n+1} = 0$, since $\mathbf{v}$ cannot be written as a linear combination of the vectors in S and $\alpha_1 = \cdots = \alpha_n = 0$, since S is linearly independent. Hence $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{v}\}$ is a linearly independent set of vectors.

But by Theorem C22, any set of more than n vectors is linearly dependent. This is a contradiction so the original statement must be false, and S does span V.

Therefore every set of n linearly independent vectors in V is a basis for V. ■

This means that to check whether a set S is a basis for $\mathbb{R}^n$, we no longer have to check that S spans $\mathbb{R}^n$: we *know* that it does if it is linearly independent and contains n vectors. We can simplify Strategy C8.

In fact, we can use this simplified strategy to determine whether a set of vectors is a basis for any vector space V *if* we know the dimension of V.

Strategy C9

To determine whether a set of vectors S in a vector space V of dimension n is a basis, check the following conditions.

(1) S contains n vectors.

(2) S is linearly independent.

If both (1) and (2) hold, then S is a basis for V.

If either (1) or (2) does not hold, then S is not a basis for V.

Exercise C64

Use Strategy C9 to determine which of the following sets is a basis for $\mathbb{R}^3$.

(a) $\{(1,2,1),(1,0,-1)\}$ (b) $\{(1,0,1),(1,0,-1),(0,1,1)\}$

(c) $\{(1,-1,0),(2,1,4),(3,0,4)\}$

(d) $\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$

Strategy C9 is easier to use than Strategy C8 because you can eliminate sets that do not contain the right number of vectors. Furthermore, you do not need to check spanning, which is usually harder than checking for linear independence.

To be able to apply Strategy C9 to vector spaces other than $\mathbb{R}^n$ we need to know the dimension of other vector spaces.

In Subsection 3.3 we listed the standard bases for some vector spaces as follows.

$$\mathbb{R}^n: \quad \{(1,0,\dots,0),(0,1,0,\dots,0),\dots,(0,\dots,0,1)\}.$$

$$P_n: \quad \{1,x,x^2,\dots,x^{n-1}\}.$$

$$M_{2,2}: \quad \left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}.$$

$$\mathbb{C}: \quad \{1,i\}.$$

We can see that the dimension of P_n is n, so the dimension of P_2 is 2, the dimension of P_3 is 3, and so on.

Similarly, the dimension of $M_{2,2}$ is 4, and, in general, the dimension of $M_{m,n}$ is mn. For example, $M_{2,3}$ has dimension 6: a basis is

$$\left\{\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}0&0&0\\1&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\end{pmatrix}\right\}.$$

Finally, the dimension of $\mathbb{C}$ is 2.

Exercise C65

Use Strategy C9 to determine whether each of the following sets is a basis for the given vector space.

(a) The set S for $M_{2,2}$, where

$$S=\left\{\begin{pmatrix}1&0\\1&0\end{pmatrix},\begin{pmatrix}0&1\\0&1\end{pmatrix},\begin{pmatrix}1&1\\0&0\end{pmatrix},\begin{pmatrix}0&1\\1&1\end{pmatrix}\right\}$$

(b) The set $S=\{2+x,1-x\}$ for P_2.

We end this section by showing that a linearly independent subset of a vector space can always be extended to give a basis for the vector space. This result will be useful in Unit C3 *Linear transformations*.

Theorem C26

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be a linearly independent subset of an n-dimensional vector space V, where $m < n$. Then there exist vectors $\mathbf{v}_{m+1}, \ldots, \mathbf{v}_n$ in V such that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V.

Proof Since $m < n$, S is not a basis for V, by the Basis Theorem (Theorem C24) and Theorem C25. Thus there is a vector $\mathbf{v}_{m+1}$ in V that cannot be expressed as a linear combination of the vectors in S. As in the proof of Theorem C25, it follows that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{m+1}\}$ is linearly independent.

We keep adding vectors in this way until we obtain a linearly independent set with n vectors. This is a basis, by Theorem C25. ■

4 Subspaces

In this section you will meet subsets of vector spaces that are themselves vector spaces.

4.1 Definition

You have seen examples where a set of vectors does not span the whole of a vector space, but spans only a proper subset of that vector space, for example in Worked Exercise C32 and Exercise C56. In particular, you saw the following.

- In $\mathbb{R}^2$, the set of vectors $\{(1, 1)\}$ is a spanning set for the line through the origin with equation $y = x$; this is a one-dimensional subset of $\mathbb{R}^2$.
- In $\mathbb{R}^3$, the set of vectors $\{(1, 0, 0)\}$ is a spanning set for the x-axis; this is a one-dimensional subset of $\mathbb{R}^3$.
- In $\mathbb{R}^3$, the set of vectors $\{(1, 0, 1), (2, 0, 3)\}$ is a spanning set for the plane $y = 0$; this is a two-dimensional subset of $\mathbb{R}^3$.

In fact, any proper subset of $\mathbb{R}^3$ that is the span of a set of vectors must take one of the following forms: $\{\mathbf{0}\}$, a line through the origin (a one-dimensional subset), or a plane through the origin (a two-dimensional subset).

When you met these examples, you may have asked yourself whether these subsets are themselves vector spaces. In fact, they are; we call such subsets *subspaces*.

Definition

A subset S of a vector space V is a **subspace** of V if S is itself a vector space under vector addition and scalar multiplication as defined in V.

In order to prove that a subset S is a vector space, we must show that it satisfies all the axioms in Subsection 1.2. In practice, however, we do not need to check them all, as many of them carry over from V; that is, if they are true for V, then they are also true for S. For example, the commutativity axiom (A5) states that $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$, for all $\mathbf{v}_1, \mathbf{v}_2 \in V$; since all the vectors in S are also in V, this axiom holds for S.

Provided that S is non-empty, the only axioms that need to be checked are the closure axioms (A1 and S1), because all the other axioms follow from V. If the zero vector is in S, then S is non-empty. Therefore we can replace the condition that S is non-empty by the condition that the zero vector is in S. This gives the following theorem; you are asked to prove this as an exercise in the additional exercises booklet for this unit.

Theorem C27

A subset S of a vector space V is a subspace of V if it satisfies the following conditions.

(a) $\mathbf{0} \in S$.

(b) S is closed under vector addition.

(c) S is closed under scalar multiplication.

This theorem allows us to give a strategy for testing whether a given subset of a vector space is a subspace.

Strategy C10

To test whether a given subset S of a vector space V is a subspace of V, check the following conditions.

(1) $\mathbf{0} \in S$ (zero vector).

(2) If $\mathbf{v}_1, \mathbf{v}_2 \in S$, then $\mathbf{v}_1 + \mathbf{v}_2 \in S$ (vector addition).

(3) If $\mathbf{v} \in S$ and $\alpha \in \mathbb{R}$, then $\alpha\mathbf{v} \in S$ (scalar multiplication).

If (1), (2) and (3) hold, then S is a subspace of V.

If any of (1), (2) or (3) does not hold, then S is not a subspace of V.

The following worked exercises and exercises illustrate how this strategy is used to show that a given set is a subspace.

Worked Exercise C42

Show that the set of vectors $S = \{(x, 3x) : x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$. Sketch this subspace.

Solution

The set S is a subset of $\mathbb{R}^2$, so we use Strategy C10.

We first check condition (1): $\mathbf{0} \in S$.

If $x = 0$, then $(x, 3x) = (0, 0)$, so S contains the zero vector of $\mathbb{R}^2$.

We check condition (2): If $\mathbf{v}_1, \mathbf{v}_2 \in S$, then $\mathbf{v}_1 + \mathbf{v}_2 \in S$.

Let $\mathbf{v}_1 = (x_1, 3x_1)$ and $\mathbf{v}_2 = (x_2, 3x_2)$ belong to S. Then

$$\begin{aligned}\mathbf{v}_1 + \mathbf{v}_2 &= (x_1, 3x_1) + (x_2, 3x_2)\\ &= (x_1 + x_2, 3x_1 + 3x_2)\\ &= (x_1 + x_2, 3(x_1 + x_2)).\end{aligned}$$

This vector has the correct form for a vector in S, since $x_1 + x_2 \in \mathbb{R}$, so S is closed under vector addition.

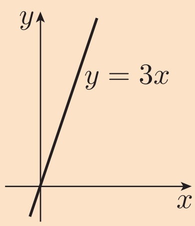
We check condition (3): If $\mathbf{v} \in S$ and $\alpha \in \mathbb{R}$, then $\alpha\mathbf{v} \in S$.

Let $\mathbf{v} = (x, 3x) \in S$ and $\alpha \in \mathbb{R}$. Then

$$\alpha\mathbf{v} = \alpha(x, 3x) = (\alpha x, \alpha 3x) = (\alpha x, 3(\alpha x)).$$

This vector has the correct form for a vector in S, since $\alpha x \in \mathbb{R}$, so S is closed under scalar multiplication.

Since conditions (1), (2) and (3) are satisfied, S is a subspace of $\mathbb{R}^2$. This subspace is the line through the origin with equation $y = 3x$.



Exercise C66

Show that the set of vectors $S = \{(x, -2x) : x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.

Worked Exercise C43

Show that the set of vectors $S = \{(x, y, 2x - 3y) : x, y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

Solution

The set S is a subset of $\mathbb{R}^3$, so we use Strategy C10.

If $x = y = 0$, then $(x, y, 2x - 3y) = (0, 0, 0)$, so S contains the zero vector of $\mathbb{R}^3$.

Let $\mathbf{v}_1 = (x_1, y_1, 2x_1 - 3y_1)$ and $\mathbf{v}_2 = (x_2, y_2, 2x_2 - 3y_2)$ belong to S. Then

$$\begin{aligned}\mathbf{v}_1 + \mathbf{v}_2 &= (x_1, y_1, 2x_1 - 3y_1) + (x_2, y_2, 2x_2 - 3y_2) \\ &= (x_1 + x_2, y_1 + y_2, 2x_1 - 3y_1 + 2x_2 - 3y_2) \\ &= (x_1 + x_2, y_1 + y_2, 2(x_1 + x_2) - 3(y_1 + y_2)).\end{aligned}$$

This vector has the correct form for a vector in S, since $x_1 + x_2, y_1 + y_2 \in \mathbb{R}$, so S is closed under vector addition.

Let $\mathbf{v} = (x, y, 2x - 3y) \in S$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}\alpha\mathbf{v} &= \alpha(x, y, 2x - 3y) \\ &= (\alpha x, \alpha y, \alpha(2x - 3y)) \\ &= (\alpha x, \alpha y, 2(\alpha x) - 3(\alpha y)).\end{aligned}$$

This vector has the correct form for a vector in S, since $\alpha x, \alpha y \in \mathbb{R}$, so S is closed under scalar multiplication.

Since conditions (1), (2) and (3) are satisfied, S is a subspace of $\mathbb{R}^3$.

S is the set of points in $\mathbb{R}^3$ satisfying $z = 2x - 3y$; it is the plane through the origin with equation $2x - 3y - z = 0$.

Strategy C10 is used in much the same way to determine whether a given subset is a subspace. However, since if any one of the conditions fails then the subset is not a subspace, it may be that only one of the conditions needs to be checked.

Worked Exercise C44

For each of the following, determine whether the set S is a subspace of the vector space $\mathbb{R}^3$.

(a) $S = \{(x, y, x - y + 2) : x, y \in \mathbb{R}\}$ (b) $S = \{(z - y, y, z) : y, z \in \mathbb{R}\}$

Solution

In each case the set S is a subset of $\mathbb{R}^3$, so we use Strategy C10.

(a) If $\mathbf{0} \in S$, then $(x, y, x - y + 2) = (0, 0, 0)$ for some numbers x and y. Equating corresponding coordinates, we obtain the system

$$\begin{aligned} x \quad\;\; &= 0 \\ y &= 0 \\ x - y &= -2. \end{aligned}$$

This system is inconsistent so has no solution. Therefore $\mathbf{0}$ does not belong to S and condition (1) is not satisfied. Hence S is not a subspace of $\mathbb{R}^3$.

Since condition (1) is not satisfied, we do not need to check conditions (2) and (3). However, neither is satisfied, and either one could have been used to show that S is not a subspace.

(b) If $y = z = 0$, then $(z - y, y, z) = (0, 0, 0)$, so S contains the zero vector of $\mathbb{R}^3$.

Let $\mathbf{v}_1 = (z_1 - y_1, y_1, z_1)$ and $\mathbf{v}_2 = (z_2 - y_2, y_2, z_2)$ belong to S. Then

$$\begin{aligned} \mathbf{v}_1 + \mathbf{v}_2 &= (z_1 - y_1, y_1, z_1) + (z_2 - y_2, y_2, z_2) \\ &= (z_1 - y_1 + z_2 - y_2, y_1 + y_2, z_1 + z_2) \\ &= ((z_1 + z_2) - (y_1 + y_2), y_1 + y_2, z_1 + z_2). \end{aligned}$$

This vector has the correct form for a vector in S, since $y_1 + y_2, z_1 + z_2 \in \mathbb{R}$, so S is closed under vector addition.

Let $\mathbf{v} = (z - y, y, z) \in S$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} \alpha\mathbf{v} &= \alpha(z - y, y, z) \\ &= (\alpha(z - y), \alpha y, \alpha z) \\ &= (\alpha z - \alpha y, \alpha y, \alpha z). \end{aligned}$$

This vector has the correct form for a vector in S, since $\alpha y, \alpha z \in \mathbb{R}$, so S is closed under scalar multiplication.

Since conditions (1), (2) and (3) are satisfied, S is a subspace of $\mathbb{R}^3$.

S is the set of points in $\mathbb{R}^3$ satisfying $z = x + y$; it is the plane through the origin with equation $x + y - z = 0$.

Exercise C67

For each of the following, determine whether the set S is a subspace of the vector space V.

(a) $V = \mathbb{R}^2$, $\quad S = \{(x, x + 2) : x \in \mathbb{R}\}$.

(b) $V = \mathbb{R}^4$, $\quad S = \{(x, y, z, x + 2y - z) : x, y, z \in \mathbb{R}\}$.

Worked Exercise C45

Determine whether the set $S = \{a\cos x : a \in \mathbb{R}\}$ is a subspace of the vector space $V = \{a\cos x + b\sin x : a, b \in \mathbb{R}\}$.

(We showed that V is a vector space in Subsection 1.2.)

Solution

The set S is a subset of V, so we use Strategy C10.

The zero vector of V is $0\cos x + 0\sin x = 0 = \mathbf{0}$. If $a = 0$, then $a\cos x = 0\cos x = 0$, so S contains the zero vector.

We need 'names' for two general vectors in S. As they are functions, we call them $f_1(x)$ and $f_2(x)$.

Let $f_1(x) = a_1\cos x$ and $f_2(x) = a_2\cos x$ belong to S. Then

$$f_1(x) + f_2(x) = a_1\cos x + a_2\cos x = (a_1 + a_2)\cos x.$$

The function $f_1(x) + f_2(x)$ has the correct form for a vector in S, since $a_1 + a_2 \in \mathbb{R}$, so S is closed under vector addition.

Let $f(x) = a\cos x \in S$ and $\alpha \in \mathbb{R}$. Then

$$\alpha f(x) = \alpha a\cos x = (\alpha a)\cos x.$$

This function has the correct form for a vector in S, since $\alpha a \in \mathbb{R}$, so S is closed under scalar multiplication.

Since conditions (1), (2) and (3) are satisfied, S is a subspace of V.

Exercise C68

For each of the following, determine whether the set S is a subspace of the vector space V.

(a) $V = P_3$, $S = \{a + bx : a, b \in \mathbb{R}\}$.

(b) $V = P_3$, $S = \{x + ax^2 : a \in \mathbb{R}\}$.

(c) $V = M_{2,2}$, $S = \left\{\begin{pmatrix} a & 1 \\ 0 & d \end{pmatrix} : a, d \in \mathbb{R}\right\}$.

The following theorem shows that the span of a subset of a vector space is always a subspace.

Theorem C28

Let S be a non-empty finite subset of a vector space V. Then $\langle S \rangle$ is a subspace of V.

Proof Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a non-empty finite subset of a vector space V. Then the set $\langle S \rangle$ is a subset of V since V is closed under vector addition and scalar multiplication.

We apply Strategy C10.

The span $\langle S \rangle$ contains the zero vector, since $0\mathbf{u}_1 + 0\mathbf{u}_2 + \cdots + 0\mathbf{u}_n = \mathbf{0}$ belongs to $\langle S \rangle$.

Let $\mathbf{v}_1 = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$ and $\mathbf{v}_2 = b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \cdots + b_n\mathbf{u}_n$ be any two vectors in $\langle S \rangle$. Then

$$\begin{aligned}\mathbf{v}_1 + \mathbf{v}_2 &= (a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n) + (b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \cdots + b_n\mathbf{u}_n) \\ &= (a_1 + b_1)\mathbf{u}_1 + (a_2 + b_2)\mathbf{u}_2 + \cdots + (a_n + b_n)\mathbf{u}_n.\end{aligned}$$

This is a member of $\langle S \rangle$, since it is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Hence $\langle S \rangle$ is closed under vector addition.

Let $\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}\alpha\mathbf{v} &= \alpha(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n) \\ &= (\alpha a_1)\mathbf{u}_1 + (\alpha a_2)\mathbf{u}_2 + \cdots + (\alpha a_n)\mathbf{u}_n.\end{aligned}$$

This is a member of $\langle S \rangle$, since it is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Hence $\langle S \rangle$ is closed under scalar multiplication.

Thus $\langle S \rangle$ is a subspace of V. ■

4.2 Bases and dimension

In the previous subsection you saw several subspaces of finite-dimensional vector spaces. Since these subspaces are all vector spaces in their own right, they have bases and dimensions, and we look at these in this subsection.

Let us return to two of our earlier examples from Section 2: Worked Exercises C32(a) and (b).

By Theorem C28, we now know that the set of vectors in $\mathbb{R}^2$ spanned by the set $S = \{(1,1)\}$ is a subspace of $\mathbb{R}^2$. In Worked Exercise C32(a) we saw that any vector in this subspace $\langle S \rangle$ can be written in the form (α, α) for some $\alpha \in \mathbb{R}$; so the set $\{(1,1)\}$ is a basis for this subspace. Thus the dimension of the subspace is 1. This agrees with our intuitive idea of dimension: we saw that these vectors form a line through the origin – the line $y = x$, as shown in Figure 11 – which is one-dimensional.

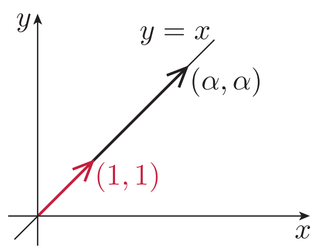


Figure 11 The one-dimensional subspace $\langle\{(1,1)\}\rangle$

Similarly, from Worked Exercise C32(b) the set of vectors in $\mathbb{R}^3$ spanned by the set $S = \{(1,0,1), (2,0,3)\}$ is a subspace of $\mathbb{R}^3$. This subspace $\langle S \rangle$ consists of those points of $\mathbb{R}^3$ of the form $(x, 0, z)$. Since the set $\{(1,0,1), (2,0,3)\}$ spans the subspace and is linearly independent (the vectors are not multiples of each other), it is a basis for this subspace. Since there are two vectors in the basis, the dimension of the subspace is 2.

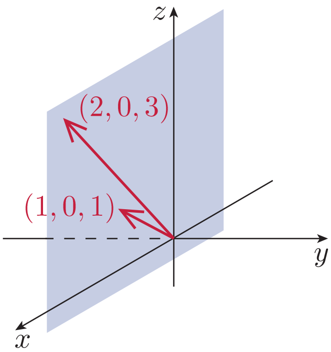


Figure 12 The two-dimensional subspace $\langle\{(1,0,1),(2,0,3)\}\rangle$

Again, this links the idea of dimension in linear algebra to our intuitive idea of dimension: we saw that the subspace spanned by these two vectors is a plane through the origin – namely, the plane $y = 0$, as shown in Figure 12 – which is two-dimensional. Since any vector in the subspace can be written in the form $(x, 0, z)$, we can find another basis for this subspace by writing

$$(x,0,z) = x(1,0,0) + z(0,0,1).$$

This means that the set $\{(1,0,0),(0,0,1)\}$ is another spanning set for the subspace and, as it is also linearly independent, it is a basis for the subspace. This basis has the additional advantage that it is *orthogonal*, which means that the basis vectors are at right angles to each other. We will return to orthogonal bases in Section 5.

In the following worked exercises and exercises we consider various subspaces of $\mathbb{R}^3$ and $\mathbb{R}^4$ and look at their bases and dimension.

Worked Exercise C46

Find the equation of the subspace of $\mathbb{R}^3$ spanned by the set $\{(1,0,2),(2,3,4)\}$.

Solution

The two vectors are not multiples of each other, so they are linearly independent.

Since $\{(1,0,2),(2,3,4)\}$ is a linearly independent set, the subspace it spans is a two-dimensional subspace of $\mathbb{R}^3$ (by Theorem C25).

A two-dimensional subspace is a plane, and since the zero vector is in the subspace this plane must pass through the origin.

The subspace is therefore a plane through the origin with equation

$$ax + by + cz = 0,$$

where a, b, c are not all zero.

Since the vectors in the spanning set lie in the plane, the values of a, b and c must satisfy the system

$$\begin{aligned} a \phantom{{}+3b} + 2c &= 0 \\ 2a + 3b + 4c &= 0. \end{aligned}$$

The first of these equations gives $a = -2c$, and substituting this into the second equation gives $b = 0$; so the subspace is the plane with equation $-2cx + cz = 0$, or, equivalently,

$$2x - z = 0.$$

Exercise C69

Find the equation of the subspace of $\mathbb{R}^3$ spanned by the set $\{(1,-2,0),(0,3,3)\}$.

Worked Exercise C47

Find a basis for the subspace $S = \{(z-y,y,z) : y,z \in \mathbb{R}\}$ of $\mathbb{R}^3$, and hence write down the dimension of S.

(You showed that S is a subspace of $\mathbb{R}^3$ in Worked Exercise C44(b).)

Solution

We use the form of the vectors in S to help us find a possible basis: the coordinates of the general vector involve y and z, so we look for vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $(z-y,y,z) = z\mathbf{v}_1 + y\mathbf{v}_2$.

Since

$$\begin{aligned}(z-y,y,z) &= (z,0,z)+(-y,y,0)\\ &= z(1,0,1)+y(-1,1,0),\end{aligned}$$

any vector in S can be written as a linear combination of the vectors in the set $\{(1,0,1),(-1,1,0)\}$, so this set spans S.

The vectors in the set are also linearly independent, as they are not multiples of each other, so $\{(1,0,1),(-1,1,0)\}$ is a basis for S.

Therefore S has dimension 2.

S is the set of points of $\mathbb{R}^3$ satisfying $x = z-y$; it is the plane with equation $x+y-z=0$.

Exercise C70

Find a basis for the subspace

$$S = \{(x,y,z,x+2y-z) : x,y,z \in \mathbb{R}\}$$

of $\mathbb{R}^4$, and hence write down the dimension of S.

(You showed that S is a subspace of $\mathbb{R}^4$ in Exercise C67(b).)

Worked Exercise C48

Find a basis for the plane $x - 3y + 2z = 0$ (a subspace of $\mathbb{R}^3$).

Solution

Since the subspace is a plane, it has dimension 2, and so has two basis vectors.

We need to find two vectors that lie in the plane and form a linearly independent set. There are many choices, so we first set $x = 0$ and then set $z = 0$ to find two vectors in the plane.

The vectors $(0, 2, 3)$ and $(3, 1, 0)$ both lie in the plane, and they are linearly independent, since one is not a multiple of the other. Therefore $\{(0, 2, 3), (3, 1, 0)\}$ is a basis for the subspace $x - 3y + 2z = 0$.

The following result, which will be used in Unit C3, has been illustrated by the worked exercises and exercises in this subsection. For example, in Worked Exercise C47 we had $V = \mathbb{R}^3$, so $\dim V = 3$ and $\dim S = 2 \leq \dim V$.

Theorem C29

The dimension of a subspace of a vector space V is less than or equal to the dimension of V.

Proof Let V be a vector space of dimension n, and let S be a subspace of V. Suppose that the dimension of S is m, and let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ be a basis for S. Then $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ is a linearly independent set of vectors in V. Thus $m \leq n$ by Corollary C23. ■

5 Orthogonal bases

In this section you will look at bases in which the basis vectors are all *orthogonal* to each other.

5.1 Orthogonal bases in $\mathbb{R}^3$

Suppose that we wish to express the vector $(10, 0, 4)$ in $\mathbb{R}^3$ in terms of the basis

$$\{(2, 1, 1), (1, -4, 2), (-2, 1, 3)\}.$$

Using the method given in Subsection 2.1, we first write

$$(10, 0, 4) = \alpha_1(2, 1, 1) + \alpha_2(1, -4, 2) + \alpha_3(-2, 1, 3).$$

Equating corresponding coordinates gives the system

$$\begin{aligned} 2\alpha_1 + \alpha_2 - 2\alpha_3 &= 10 \\ \alpha_1 - 4\alpha_2 + \alpha_3 &= 0 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 &= 4. \end{aligned}$$

We can solve this system using Gauss–Jordan elimination or directly, to obtain the solution

$$\alpha_1 = 4, \quad \alpha_2 = \tfrac{6}{7}, \quad \alpha_3 = -\tfrac{4}{7}.$$

Thus

$$(10, 0, 4) = 4(2, 1, 1) + \tfrac{6}{7}(1, -4, 2) - \tfrac{4}{7}(-2, 1, 3).$$

In this section you will see that there is a simpler method than this that involves scalar products of vectors. It can be used when, as here, the given basis is an *orthogonal* basis. In this subsection we concentrate on $\mathbb{R}^3$.

We start by recalling from Unit A1 the definition of the scalar product in $\mathbb{R}^3$, and then use this to define the term *orthogonal*.

Definitions

Let $\mathbf{v}_1 = (x_1, y_1, z_1)$ and $\mathbf{v}_2 = (x_2, y_2, z_2)$ be vectors in $\mathbb{R}^3$.

The **scalar product** of $\mathbf{v}_1$ and $\mathbf{v}_2$ is the real number

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1x_2 + y_1y_2 + z_1z_2.$$

The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^3$ are **orthogonal** if $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.

For example, the vectors $\mathbf{v}_1 = (2, 1, 1)$ and $\mathbf{v}_2 = (-2, 1, 3)$ are orthogonal, since

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 2 \times (-2) + 1 \times 1 + 1 \times 3 = -4 + 1 + 3 = 0.$$

Geometrically, this means that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are at right angles to each other, as shown in Figure 13.

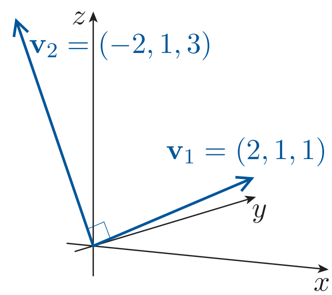


Figure 13 The orthogonal vectors $\mathbf{v}_1 = (2, 1, 1)$ and $\mathbf{v}_2 = (-2, 1, 3)$

Exercise C71

(a) Show that $(2, 1, 1)$ and $(1, -4, 2)$ are orthogonal.

(b) Determine which pairs of the following vectors are orthogonal:

$$\mathbf{v}_1 = (-2, 6, 1), \quad \mathbf{v}_2 = (9, 2, 6), \quad \mathbf{v}_3 = (4, -15, -1).$$

Definition

A set of vectors in $\mathbb{R}^3$ is an **orthogonal set** if every pair of distinct vectors in the set is orthogonal.

For example, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal set if $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$; we have therefore shown above that $\{(2,1,1), (1,-4,2)\}$ is an orthogonal set.

Similarly, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set if

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 0.$$

So $\{(2,1,1), (1,-4,2), (-2,1,3)\}$ is an orthogonal set since

$$(1,-4,2) \cdot (-2,1,3) = -2 - 4 + 6 = 0,$$

and we have shown that $(2,1,1) \cdot (1,-4,2) = 0$ and $(2,1,1) \cdot (-2,1,3) = 0$.

One of the most useful features of orthogonal sets of non-zero vectors is their linear independence. The following proof is for sets of three non-zero vectors, but a similar proof applies to other numbers of vectors and indeed to orthogonal sets of vectors in $\mathbb{R}^n$.

Theorem C30

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthogonal set of non-zero vectors in $\mathbb{R}^3$. Then $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent.

Proof To show that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent we need to deduce that if $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}$ then $\alpha_1 = \alpha_2 = \alpha_3 = 0$ by using the properties of scalar products.

Suppose that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}.$$

We form the scalar product on both sides of the equation with $\mathbf{v}_1$:

$$\mathbf{v}_1 \cdot (\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3) = \mathbf{v}_1 \cdot \mathbf{0} = 0.$$

Using the multiples property of the scalar product (Unit A1) we get

$$\alpha_1(\mathbf{v}_1 \cdot \mathbf{v}_1) + \alpha_2(\mathbf{v}_1 \cdot \mathbf{v}_2) + \alpha_3(\mathbf{v}_1 \cdot \mathbf{v}_3) = 0.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set of non-zero vectors in $\mathbb{R}^3$, we know that

$$\mathbf{v}_1 \cdot \mathbf{v}_1 \neq 0,\ \mathbf{v}_1 \cdot \mathbf{v}_2 = 0,\ \mathbf{v}_1 \cdot \mathbf{v}_3 = 0,$$

so we have $\alpha_1(\mathbf{v}_1 \cdot \mathbf{v}_1) = 0$ and thus $\alpha_1 = 0$.

Similarly, we form the scalar product with $\mathbf{v}_2$ and $\mathbf{v}_3$:

$$\mathbf{v}_2 \cdot (\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3) = \mathbf{v}_2 \cdot \mathbf{0} = 0,$$

which gives $\alpha_2 = 0$;

$$\mathbf{v}_3 \cdot (\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3) = \mathbf{v}_3 \cdot \mathbf{0} = 0,$$

which gives $\alpha_3 = 0$.

We conclude that if $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}$ then $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set. ■

This result leads to the idea of an *orthogonal basis*.

You have seen that any linearly independent set of three vectors in $\mathbb{R}^3$ is a basis for $\mathbb{R}^3$. Now, if we have an orthogonal set of three non-zero vectors in $\mathbb{R}^3$, then we know from Theorem C30 that the set is linearly independent, so the set is a basis for $\mathbb{R}^3$. We call an orthogonal set that is a basis an **orthogonal basis**.

Theorem C31

Any orthogonal set of three non-zero vectors in $\mathbb{R}^3$ is an orthogonal basis for $\mathbb{R}^3$.

For example, the standard basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ for $\mathbb{R}^3$ is an orthogonal basis, because these three basis vectors form an orthogonal set. Similarly, the triple of vectors below is an orthogonal basis for $\mathbb{R}^3$ since the vectors are orthogonal (as we saw above), there are three of them, and they are all non-zero:

$$\{(2,1,1),(1,-4,2),(-2,1,3)\}.$$

One reason that orthogonal bases are so important is that it is usually much easier to express a vector in terms of an orthogonal basis than in terms of a general basis. At the beginning of this subsection we expressed $(10,0,4)$ in terms of the orthogonal basis $\{(2,1,1),(1,-4,2),(-2,1,3)\}$ by writing

$$(10,0,4)=\alpha_1(2,1,1)+\alpha_2(1,-4,2)+\alpha_3(-2,1,3) \qquad (3)$$

and solving the resulting system of linear equations.

However, there is a quicker way of solving equation (3) because the basis is an orthogonal basis. We take the scalar product of the vector $(10,0,4)$ expressed as in equation (3) with each basis vector in turn, making use of the fact that the scalar product of orthogonal vectors is zero.

First with $(2,1,1)$:

$$\begin{aligned}(2,1,1)\cdot(10,0,4)&=\alpha_1(2,1,1)\cdot(2,1,1)+\alpha_2(2,1,1)\cdot(1,-4,2)\\&\quad+\alpha_3(2,1,1)\cdot(-2,1,3)\\&=\alpha_1(2,1,1)\cdot(2,1,1)+0+0.\end{aligned}$$

The equation above gives

$$\alpha_1=\frac{(2,1,1)\cdot(10,0,4)}{(2,1,1)\cdot(2,1,1)}=\frac{24}{6}=4.$$

Similarly, taking the scalar product with $(1,-4,2)$:

$$(1,-4,2)\cdot(10,0,4)=0+\alpha_2(1,-4,2)\cdot(1,-4,2)+0.$$

Thus

$$\alpha_2=\frac{(1,-4,2)\cdot(10,0,4)}{(1,-4,2)\cdot(1,-4,2)}=\frac{18}{21}=\frac{6}{7}.$$

Finally, taking the scalar product with $(-2,1,3)$:

$$(-2,1,3)\cdot(10,0,4) = 0 + 0 + \alpha_3(-2,1,3)\cdot(-2,1,3).$$

Thus

$$\alpha_3 = \frac{(-2,1,3)\cdot(10,0,4)}{(-2,1,3)\cdot(-2,1,3)} = \frac{-8}{14} = -\frac{4}{7}.$$

Therefore, we have $\alpha_1 = 4$, $\alpha_2 = \frac{6}{7}$ and $\alpha_3 = -\frac{4}{7}$, so

$$(10,0,4) = 4(2,1,1) + \tfrac{6}{7}(1,-4,2) - \tfrac{4}{7}(-2,1,3).$$

This procedure works for orthogonal bases in general in $\mathbb{R}^3$ and is summarised in the following strategy.

Strategy C11

To express a vector $\mathbf{u}$ in $\mathbb{R}^3$ in terms of an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$:

1. calculate $\alpha_1 = \dfrac{\mathbf{v}_1\cdot\mathbf{u}}{\mathbf{v}_1\cdot\mathbf{v}_1}$, $\alpha_2 = \dfrac{\mathbf{v}_2\cdot\mathbf{u}}{\mathbf{v}_2\cdot\mathbf{v}_2}$ and $\alpha_3 = \dfrac{\mathbf{v}_3\cdot\mathbf{u}}{\mathbf{v}_3\cdot\mathbf{v}_3}$
2. write $\mathbf{u} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3$.

Exercise C72

(a) Verify that $\{(3,4,0), (8,-6,0), (0,0,5)\}$ is an orthogonal basis for $\mathbb{R}^3$.

(b) Express the vector $(10,0,4)$ in terms of this basis.

5.2 Orthogonal bases in $\mathbb{R}^n$

In this subsection we see how the definitions and results of the previous subsection can be generalised to $\mathbb{R}^n$, for any positive integer n. We start with the definition of the scalar product of vectors.

Definition

Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be vectors in $\mathbb{R}^n$. The **scalar product** of $\mathbf{v}$ and $\mathbf{w}$ is the real number

$$\mathbf{v}\cdot\mathbf{w} = v_1w_1 + v_2w_2 + \cdots + v_nw_n.$$

For example, in $\mathbb{R}^5$ the scalar product of the vectors $\mathbf{v} = (1,2,3,4,5)$ and $\mathbf{w} = (3,-4,0,3,-2)$ is

$$\begin{aligned}\mathbf{v}\cdot\mathbf{w} &= 1\times 3 + 2\times(-4) + 3\times 0 + 4\times 3 + 5\times(-2)\\ &= 3 - 8 + 0 + 12 - 10 = -3.\end{aligned}$$

Exercise C73

Calculate the following scalar products.

(a) $(1, 2, -1, 0) \cdot (0, -5, 6, -3)$ in $\mathbb{R}^4$.

(b) $(1, 2, 3, 4, 5, 6) \cdot (3, 2, 1, 0, -1, -2)$ in $\mathbb{R}^6$.

We now see how the ideas of an orthogonal set and an orthogonal basis extend to $\mathbb{R}^n$.

Definitions

The vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ are **orthogonal** if $\mathbf{v} \cdot \mathbf{w} = 0$.

A set of vectors in $\mathbb{R}^n$ is an **orthogonal set** if every pair of distinct vectors in the set is orthogonal.

An **orthogonal basis** for $\mathbb{R}^n$ is an orthogonal set that is a basis for $\mathbb{R}^n$.

For example, in $\mathbb{R}^6$ the set

$$\{(1, 1, 1, 1, 1, 1), (2, -2, 2, -2, 2, -2), (5, 5, 0, 0, -5, -5)\}$$

is an orthogonal set, since

$$\begin{aligned}(1, 1, 1, 1, 1, 1) &\cdot (2, -2, 2, -2, 2, -2)\\ &= 2 - 2 + 2 - 2 + 2 - 2 = 0,\end{aligned}$$

$$\begin{aligned}(1, 1, 1, 1, 1, 1) &\cdot (5, 5, 0, 0, -5, -5)\\ &= 5 + 5 + 0 + 0 - 5 - 5 = 0\end{aligned}$$

and

$$\begin{aligned}(2, -2, 2, -2, 2, -2) &\cdot (5, 5, 0, 0, -5, -5)\\ &= 10 - 10 + 0 + 0 - 10 + 10 = 0.\end{aligned}$$

Exercise C74

Show that the set $\{(1, 0, 0, 0, 0), (0, 2, 0, 0, 0), (0, 0, 1, 1, 0)\}$ is an orthogonal set in $\mathbb{R}^5$.

Note that the standard basis

$$\{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\}$$

is an orthogonal basis for $\mathbb{R}^n$.

In Subsection 5.1 you saw that any orthogonal set of three non-zero vectors in $\mathbb{R}^3$ is linearly independent and therefore forms an orthogonal basis for $\mathbb{R}^3$. Exactly the same methods can be used to prove the following more general result.

Theorem C32

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be an orthogonal set of non-zero vectors in $\mathbb{R}^n$. Then S is a linearly independent set.

Since any set of n linearly independent vectors in $\mathbb{R}^n$ forms a basis for $\mathbb{R}^n$, we obtain the following corollary to Theorem C32.

Corollary C33

Any orthogonal set of n non-zero vectors in $\mathbb{R}^n$ is an orthogonal basis for $\mathbb{R}^n$.

Exercise C75

Show that

$$\{(1,2,1,0), (-1,1,-1,1), (1,0,-1,0), (1,-1,1,3)\}$$

is an orthogonal basis for $\mathbb{R}^4$.

Expressing vectors in terms of orthogonal bases

Given an orthogonal basis for $\mathbb{R}^n$, it is particularly easy to express any given vector as a linear combination of the basis vectors. As for $\mathbb{R}^3$ in Subsection 5.1, we simply need to calculate scalar products: we do not need to solve a system of linear equations.

Theorem C34

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthogonal basis for $\mathbb{R}^n$ and let $\mathbf{u}$ be any vector in $\mathbb{R}^n$. Then

$$\mathbf{u} = \left(\frac{\mathbf{v}_1 \cdot \mathbf{u}}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 + \left(\frac{\mathbf{v}_2 \cdot \mathbf{u}}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 + \cdots + \left(\frac{\mathbf{v}_n \cdot \mathbf{u}}{\mathbf{v}_n \cdot \mathbf{v}_n}\right)\mathbf{v}_n.$$

Proof Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthogonal basis for $\mathbb{R}^n$ and let $\mathbf{u}$ be any vector in $\mathbb{R}^n$. Since $\mathbf{u} \in \mathbb{R}^n$, we can write $\mathbf{u}$ as a linear combination of the basis vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$:

$$\mathbf{u} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n. \tag{4}$$

Forming the scalar product of both sides of equation (4) with $\mathbf{v}_1$ gives

$$\mathbf{v}_1 \cdot \mathbf{u} = \alpha_1(\mathbf{v}_1 \cdot \mathbf{v}_1) \quad \text{(all other terms are 0)},$$

so $\alpha_1 = \dfrac{\mathbf{v}_1 \cdot \mathbf{u}}{\mathbf{v}_1 \cdot \mathbf{v}_1}$.

Similarly, forming the scalar product of both sides of equation (4) with $\mathbf{v}_2$ gives

$$\mathbf{v}_2 \cdot \mathbf{u} = \alpha_2(\mathbf{v}_2 \cdot \mathbf{v}_2) \quad \text{(all other terms are 0)},$$

so $\alpha_2 = \dfrac{\mathbf{v}_2 \cdot \mathbf{u}}{\mathbf{v}_2 \cdot \mathbf{v}_2}$.

Continuing in this way, we deduce that

$$\alpha_i = \frac{\mathbf{v}_i \cdot \mathbf{u}}{\mathbf{v}_i \cdot \mathbf{v}_i} \quad \text{for each } i = 1, 2, \ldots, n.$$

Thus

$$\mathbf{u} = \left(\frac{\mathbf{v}_1 \cdot \mathbf{u}}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 + \left(\frac{\mathbf{v}_2 \cdot \mathbf{u}}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 + \cdots + \left(\frac{\mathbf{v}_n \cdot \mathbf{u}}{\mathbf{v}_n \cdot \mathbf{v}_n}\right)\mathbf{v}_n,$$

as required. ■

The result of Theorem C34 can be expressed in the form of a strategy that generalises Strategy C11.

Strategy C12

To express a vector $\mathbf{u}$ in $\mathbb{R}^n$ in terms of an orthogonal basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$:

1. calculate $\alpha_1 = \dfrac{\mathbf{v}_1 \cdot \mathbf{u}}{\mathbf{v}_1 \cdot \mathbf{v}_1}, \alpha_2 = \dfrac{\mathbf{v}_2 \cdot \mathbf{u}}{\mathbf{v}_2 \cdot \mathbf{v}_2}, \ldots, \alpha_n = \dfrac{\mathbf{v}_n \cdot \mathbf{u}}{\mathbf{v}_n \cdot \mathbf{v}_n}$
2. write $\mathbf{u} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$.

Exercise C76

Express the vector $(1, 2, 3, 4)$ in terms of the orthogonal basis for $\mathbb{R}^4$ $\{(1, 2, 1, 0), (-1, 1, -1, 1), (1, 0, -1, 0), (1, -1, 1, 3)\}$.

(You showed that this basis is orthogonal in Exercise C75.)

5.3 Constructing orthogonal bases

We now consider how to find an orthogonal basis.

Suppose we want to find an orthogonal basis for $\mathbb{R}^3$ containing the vector $(2, 1, 1)$. This means that we need to find two more vectors orthogonal to each other and orthogonal to the vector $(2, 1, 1)$.

Now recall from Unit A1 that in $\mathbb{R}^3$ a vector normal to a plane is perpendicular (orthogonal) to every vector in this plane. Thus to find such a pair of vectors, we can find two orthogonal vectors in the plane through the origin that has normal vector $(2, 1, 1)$.

Using the vector equation of a plane from Unit A1, the vector equation of a plane through the origin with normal vector $\mathbf{n}$ is

$$\mathbf{x} \cdot \mathbf{n} = 0,$$

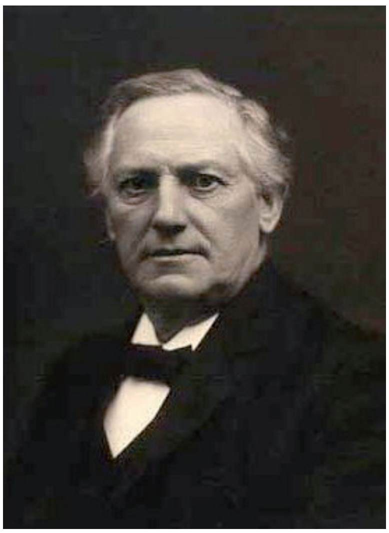
so here we have $(x, y, z) \cdot (2, 1, 1) = 0$; that is, the equation of the plane is

$$2x + y + z = 0.$$

Rather than pulling two *orthogonal* vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in this plane out of a hat, we start with *any* pair of linearly independent vectors in this plane and follow a method known as the *Gram–Schmidt orthogonalisation process* to construct a pair of orthogonal vectors.

Erhard Schmidt

In 1907, the German mathematician Erhard Schmidt (1876–1959) published an orthogonalisation algorithm, which became widely used. Schmidt acknowledged that his process was essentially the same as that published by the Danish mathematician Jørgen Pedersen Gram (1850–1916) in 1883. It appears that their names were first linked together in the 1930s. A related algorithm (now known as *modified* Gram–Schmidt) had been used much earlier by the French mathematician and scientist Pierre-Simon Laplace (1749–1827) in an attempt to estimate the masses of Jupiter and Saturn using the astronomical data of six planets.



Jørgen Pedersen Gram

To find a pair of linearly independent vectors in the plane $2x + y + z = 0$, we need to find any two vectors in this plane that are not multiples of one another. We choose suitable vectors that are as simple as possible, for example, ones containing small numbers and zeros. We start by setting x to 1 and then setting z and y to 0 in turn, to get a pair of vectors. This gives

$$\mathbf{w}_1 = (1, -2, 0) \quad \text{and} \quad \mathbf{w}_2 = (1, 0, -2).$$

Since these vectors are linearly independent, the set $\{\mathbf{w}_1, \mathbf{w}_2\}$ forms a basis for this plane. (Any other pair of linearly independent vectors in the plane would do just as well.)

We take the first vector $\mathbf{v}_1$ in our orthogonal basis to be the first of these vectors, so

$$\mathbf{v}_1 = \mathbf{w}_1 = (1, -2, 0).$$

For the second vector $\mathbf{v}_2$ in our orthogonal basis, we start with $\mathbf{w}_2$ and then subtract from it a suitable multiple α of $\mathbf{v}_1$, chosen so that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal, as illustrated in Figure 14. Since $\mathbf{v}_2$ is a linear combination of vectors in the plane and the plane is a subspace, we know that $\mathbf{v}_2$ is also in the plane.

So we set

$$\mathbf{v}_2 = \mathbf{w}_2 - \alpha \mathbf{v}_1;$$

that is,

$$\mathbf{v}_2 = (1, 0, -2) - \alpha(1, -2, 0).$$

We want to find the value of α so that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. Therefore we must have

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_2 &= \mathbf{v}_1 \cdot (\mathbf{w}_2 - \alpha \mathbf{v}_1) \\ &= \mathbf{v}_1 \cdot \mathbf{w}_2 - \alpha \mathbf{v}_1 \cdot \mathbf{v}_1 \\ &= 0. \end{aligned}$$

Hence

$$\alpha = \frac{\mathbf{v}_1 \cdot \mathbf{w}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1};$$

that is, in this case

$$\alpha = \frac{(1, -2, 0) \cdot (1, 0, -2)}{(1, -2, 0) \cdot (1, -2, 0)} = \frac{1}{5}.$$

Thus

$$\mathbf{v}_2 = (1, 0, -2) - \tfrac{1}{5}(1, -2, 0) = \left(\tfrac{4}{5}, \tfrac{2}{5}, -2\right).$$

So an orthogonal basis for the plane is $\left\{(1, -2, 0), \left(\tfrac{4}{5}, \tfrac{2}{5}, -2\right)\right\}$.

Returning to the original problem, this means that we have found that an orthogonal basis for $\mathbb{R}^3$ containing the vector $(2, 1, 1)$ is

$$\left\{(2, 1, 1), (1, -2, 0), \left(\tfrac{4}{5}, \tfrac{2}{5}, -2\right)\right\}.$$

The next exercise asks you to find an orthogonal basis for $\mathbb{R}^3$ containing a given vector by using the above method.

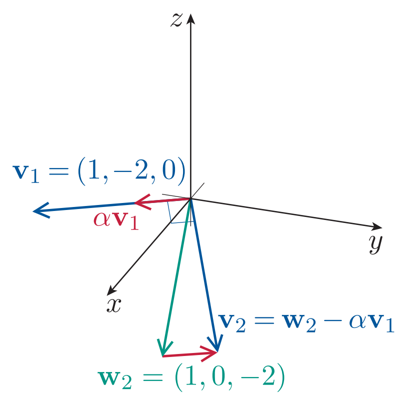


Figure 14 Subtracting a bit of $\mathbf{v}_1$ from $\mathbf{w}_2$ to get an orthogonal vector

Exercise C77

(a) Find the equation of the plane through the origin with normal vector $\mathbf{n} = (3, -4, 5)$.

(b) Show that the vectors $\mathbf{w}_1 = (4, 3, 0)$ and $\mathbf{w}_2 = (0, 5, 4)$ lie in this plane.

(c) Find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for the plane where $\mathbf{v}_1 = \mathbf{w}_1$, and

$$\mathbf{v}_2 = \mathbf{w}_2 - \frac{\mathbf{v}_1 \cdot \mathbf{w}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1.$$

(d) Hence write down an orthogonal basis for $\mathbb{R}^3$ containing the vector $(3, -4, 5)$.

In these examples we started with a pair of arbitrary basis vectors for a plane and adjusted the second to obtain a pair of orthogonal basis vectors. This method can be extended to higher-dimensional spaces by starting with an arbitrary basis and adjusting the basis vectors one by one to obtain an orthogonal basis. It is called the *Gram–Schmidt orthogonalisation process*.

Theorem C35 Gram–Schmidt orthogonalisation process

Let $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ be a basis for $\mathbb{R}^n$, and let

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{w}_1,\\
\mathbf{v}_2 &= \mathbf{w}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1,\\
\mathbf{v}_3 &= \mathbf{w}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2,\\
&\vdots\\
\mathbf{v}_n &= \mathbf{w}_n - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_n}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_n}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2\\
&\quad - \cdots - \left(\frac{\mathbf{v}_{n-1} \cdot \mathbf{w}_n}{\mathbf{v}_{n-1} \cdot \mathbf{v}_{n-1}}\right)\mathbf{v}_{n-1}.
\end{aligned}$$

Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for $\mathbb{R}^n$.

Proof We show that each vector in the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is orthogonal to every other vector in the set.

We note first that $\mathbf{v}_2$ is orthogonal to $\mathbf{v}_1$, since

$$\begin{aligned}
\mathbf{v}_1 \cdot \mathbf{v}_2 &= \mathbf{v}_1 \cdot \left(\mathbf{w}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1\right)\\
&= (\mathbf{v}_1 \cdot \mathbf{w}_2) - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)(\mathbf{v}_1 \cdot \mathbf{v}_1)\\
&= (\mathbf{v}_1 \cdot \mathbf{w}_2) - (\mathbf{v}_1 \cdot \mathbf{w}_2) = 0.
\end{aligned}$$

Next we note that $\mathbf{v}_3$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, since

$$\begin{aligned}
\mathbf{v}_1 \cdot \mathbf{v}_3 &= \mathbf{v}_1 \cdot \left(\mathbf{w}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2\right) \\
&= (\mathbf{v}_1 \cdot \mathbf{w}_3) - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)(\mathbf{v}_1 \cdot \mathbf{v}_1) - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)(\mathbf{v}_1 \cdot \mathbf{v}_2) \\
&= (\mathbf{v}_1 \cdot \mathbf{w}_3) - (\mathbf{v}_1 \cdot \mathbf{w}_3) - 0 \\
&= 0
\end{aligned}$$

because $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.

Similarly,

$$\begin{aligned}
\mathbf{v}_2 \cdot \mathbf{v}_3 &= \mathbf{v}_2 \cdot \left(\mathbf{w}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2\right) \\
&= (\mathbf{v}_2 \cdot \mathbf{w}_3) - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)(\mathbf{v}_2 \cdot \mathbf{v}_1) - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)(\mathbf{v}_2 \cdot \mathbf{v}_2) \\
&= (\mathbf{v}_2 \cdot \mathbf{w}_3) - 0 - (\mathbf{v}_2 \cdot \mathbf{w}_3) \\
&= 0.
\end{aligned}$$

Continuing in this way, we deduce that each of the vectors $\mathbf{v}_i$ is orthogonal to all the previous ones. It follows that $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all i, j with $i \neq j$, and hence that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for $\mathbb{R}^n$. ■

Exercise C78

Apply the Gram–Schmidt orthogonalisation process to the following basis for $\mathbb{R}^5$:

$$\{(1,0,0,0,0),(0,2,0,0,0),(0,0,1,1,0),(1,1,1,1,1),(1,0,-1,0,1)\}.$$

(You showed, in Exercise C74, that $\{(1,0,0,0,0),(0,2,0,0,0),(0,0,1,1,0)\}$ is an orthogonal set in $\mathbb{R}^5$.)

5.4 Orthonormal bases

You have seen that using orthogonal basis vectors can be helpful. However, in many examples it is also useful to require one further condition – that the basis vectors are all unit vectors, as in the standard basis for $\mathbb{R}^n$.

Recall, from Unit A1, that the magnitude of a vector $\mathbf{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is

$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

For example, if $\mathbf{v} = (5,-12)$, then $|\mathbf{v}| = \sqrt{5^2 + (-12)^2} = \sqrt{169} = 13$, as illustrated in Figure 15.

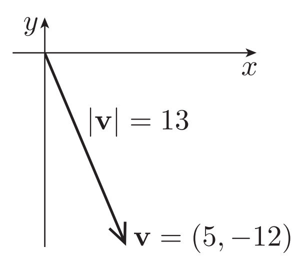


Figure 15 The magnitude of the vector $(5,-12)$

We can similarly define the magnitude of a vector in $\mathbb{R}^n$, for any positive integer n.

Definition

Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be a vector in $\mathbb{R}^n$. Then the **magnitude** of $\mathbf{v}$ is

$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

Exercise C79

Calculate the magnitude of each of the following vectors.

(a) $(3, -4, 5)$ in $\mathbb{R}^3$. (b) $(1, 2, -1, 0, 3)$ in $\mathbb{R}^5$.

Exercise C80

Prove that if $\mathbf{v}$ is any non-zero vector in $\mathbb{R}^n$, then the vector

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{|\mathbf{v}|}\mathbf{v}$$

has magnitude 1.

We make the following important definition.

Definition

An **orthonormal basis** for $\mathbb{R}^n$ is an orthogonal basis in which each basis vector has magnitude 1.

An orthonormal basis is therefore comprised of orthogonal unit vectors.

It follows from the result of Exercise C80 that, given an orthogonal basis for $\mathbb{R}^n$, we can obtain an orthonormal basis by scalar multiplication: we need to multiply each basis vector by the reciprocal of its magnitude. This leads to the following strategy for constructing an orthonormal basis.

Strategy C13

To construct an orthonormal basis for $\mathbb{R}^n$ from an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for $\mathbb{R}^n$:

1. calculate the magnitude of each basis vector
2. scalar multiply each basis vector by the reciprocal of its magnitude.

The required orthonormal basis is $\left\{\dfrac{\mathbf{v}_1}{|\mathbf{v}_1|}, \dfrac{\mathbf{v}_2}{|\mathbf{v}_2|}, \ldots, \dfrac{\mathbf{v}_n}{|\mathbf{v}_n|}\right\}$.

As a shorthand for 'scalar multiply a vector by the reciprocal of its magnitude', we may say 'divide a vector by its magnitude'.

For example, we can use Strategy C13 to obtain an orthonormal basis for $\mathbb{R}^3$ starting with the orthogonal basis $\{(2,1,1),(1,-4,2),(-2,1,3)\}$, as follows. We calculate the magnitude of each basis vector:

$$|(2,1,1)| = \sqrt{2^2+1^2+1^2} = \sqrt{6},$$
$$|(1,-4,2)| = \sqrt{1^2+(-4)^2+2^2} = \sqrt{21},$$
$$|(-2,1,3)| = \sqrt{(-2)^2+1^2+3^2} = \sqrt{14}.$$

Dividing each orthogonal basis vector by its magnitude, we obtain the orthonormal basis

$$\left\{\frac{1}{\sqrt{6}}(2,1,1), \frac{1}{\sqrt{21}}(1,-4,2), \frac{1}{\sqrt{14}}(-2,1,3)\right\}.$$

Exercise C81

Construct an orthonormal basis for $\mathbb{R}^4$, starting with the basis

$$\{(1,2,1,0),(-1,1,-1,1),(1,0,-1,0),(1,-1,1,3)\}.$$

(You showed, in Exercise C75, that this is an orthogonal basis for $\mathbb{R}^4$.)

Note that some of our earlier results become much simpler if we use an orthonormal basis, rather than an orthogonal one. For example, Theorem C34 takes the following form because $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ for each $i \leq n$.

Theorem C36

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal basis for $\mathbb{R}^n$, and let $\mathbf{u}$ be any vector in $\mathbb{R}^n$. Then

$$\mathbf{u} = (\mathbf{v}_1 \cdot \mathbf{u})\mathbf{v}_1 + (\mathbf{v}_2 \cdot \mathbf{u})\mathbf{v}_2 + \cdots + (\mathbf{v}_n \cdot \mathbf{u})\mathbf{v}_n.$$

5.5 Other vector spaces

We conclude this section by remarking that it is possible to define scalar products in vector spaces other than $\mathbb{R}^n$. For example, in the vector space P_3 we can define the scalar product of two polynomials p_1 and p_2 by

$$p_1 \cdot p_2 = \int_{-1}^{1} p_1(x)p_2(x)\,dx.$$

Such a scalar product is a real number and has properties that are very similar to those of the scalar product in $\mathbb{R}^n$ – for example, $p_1 \cdot p_2 = p_2 \cdot p_1$ for any polynomials p_1 and p_2.

We can then define such concepts as *orthogonal polynomials*, the *magnitude of a polynomial*, and the *distance and angle between two polynomials*. For example, the polynomials $p_1(x) = x$ and $p_2(x) = x^2$ are orthogonal, since

$$p_1 \cdot p_2 = \int_{-1}^{1} x \cdot x^2 \, dx = \left[\tfrac{1}{4}x^4\right]_{-1}^{1} = 0$$

and the magnitude of p_2 is given by

$$|p_2|^2 = p_2 \cdot p_2 = \int_{-1}^{1} x^2 \cdot x^2 \, dx = \left[\tfrac{1}{5}x^5\right]_{-1}^{1} = \tfrac{2}{5},$$

so $|p_2| = \sqrt{\tfrac{2}{5}}$.

Although such concepts may seem at first sight to make little sense intuitively, they have proved to be of great interest and importance, for example in mathematical physics. They also show that the mathematical structures we have introduced theoretically here can have surprising applications in other contexts.

Summary

In this unit you have seen how familiar properties of $\mathbb{R}^2$ and $\mathbb{R}^3$ can be generalised to other, very different sets of *vectors* through the concept of a *vector space*.

Your study of vector spaces has been driven by looking at properties of $\mathbb{R}^2$ and $\mathbb{R}^3$, such as linear combinations, linear independence and spanning sets of vectors. You have seen how the familiar concept of axes and our intuitive idea of dimension relate to bases of these spaces. You have seen how these concepts generalise to $\mathbb{R}^n$ and other, very different vector spaces such as P_n, $M_{m,n}$ and $\mathbb{C}$. You have met the Basis Theorem, which states that every basis for a given vector space has the same number of vectors, and that this number is the dimension of the vector space.

Starting with subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$ that can be visualised geometrically, you have seen that subspaces of vector spaces are subsets that are themselves vector spaces, in the same way that subgroups are subsets of groups that are themselves groups.

Finally, you have seen how the scalar product and orthogonality of vectors in $\mathbb{R}^n$ can be used to find orthogonal and orthonormal bases, which are particularly straightforward to work with.

Vector spaces will underpin the remainder of the linear algebra units; in particular you will study functions between vector spaces in Unit C3 *Linear transformations* and use orthonormal bases to classify conics and quadrics in Unit C4 *Eigenvectors*.

Learning outcomes

After working through this unit, you should be able to:

- understand the definition of a *real vector space*
- check whether or not a given set of elements forms a vector space under the operations of vector addition and scalar multiplication
- explain the meaning of the terms *linear combination*, *span* and *spanning set*
- form linear combinations of vectors in a given set
- check whether a vector can be expressed as a linear combination of given vectors
- find the set spanned by a given set of vectors
- check whether a given set of vectors spans the vector space to which the vectors belong
- explain the meaning of the terms *linear independence*, *linear dependence*, *basis* and *dimension*
- test whether a given set of vectors is linearly independent
- test whether a given set of vectors is a basis for a given vector space
- find the E-coordinate representation of a vector given in standard coordinates, and vice versa
- explain what is meant by a *subspace* of a vector space
- test whether a given subset of a vector space is a subspace
- find a basis for a subspace, and hence find its dimension
- check whether the vectors in a given set are *orthogonal*
- express a given vector in terms of an *orthogonal basis*
- use the Gram–Schmidt orthogonalisation process to find orthogonal bases in $\mathbb{R}^n$
- given an orthogonal basis, construct an orthonormal basis.

Solutions to exercises

Solution to Exercise C44

$$\begin{aligned}\mathbf{u}+\mathbf{v} &= (1,-1,2,0,-3)+(0,2,-1,4,0)\\ &= (1,1,1,4,-3)\end{aligned}$$

$$\begin{aligned}-3\mathbf{u} &= -3(1,-1,2,0,-3)\\ &= (-3,3,-6,0,9)\end{aligned}$$

Solution to Exercise C45

Let $\mathbf{u}=(u_1,u_2,u_3,u_4)$, $\mathbf{v}=(v_1,v_2,v_3,v_4)$ and $\mathbf{w}=(w_1,w_2,w_3,w_4)$.

(a) $(\mathbf{u}+\mathbf{v})+\mathbf{w}$

$$\begin{aligned}&= ((u_1,u_2,u_3,u_4)+(v_1,v_2,v_3,v_4))\\ &\quad +(w_1,w_2,w_3,w_4)\\ &= (u_1+v_1,u_2+v_2,u_3+v_3,u_4+v_4)\\ &\quad +(w_1,w_2,w_3,w_4)\\ &= (u_1+v_1+w_1,u_2+v_2+w_2,u_3+v_3+w_3,\\ &\qquad u_4+v_4+w_4),\end{aligned}$$

$\mathbf{u}+(\mathbf{v}+\mathbf{w})$

$$\begin{aligned}&= (u_1,u_2,u_3,u_4)\\ &\quad +((v_1,v_2,v_3,v_4)+(w_1,w_2,w_3,w_4))\\ &= (u_1,u_2,u_3,u_4)\\ &\quad +(v_1+w_1,v_2+w_2,v_3+w_3,v_4+w_4)\\ &= (u_1+v_1+w_1,u_2+v_2+w_2,u_3+v_3+w_3,\\ &\qquad u_4+v_4+w_4).\end{aligned}$$

Therefore $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$, and so the associative property (A2) holds.

(b) $\mathbf{v}+(-\mathbf{v})$

$$\begin{aligned}&= (v_1,v_2,v_3,v_4)+(-v_1,-v_2,-v_3,-v_4)\\ &= (v_1-v_1,v_2-v_2,v_3-v_3,v_4-v_4)\\ &= (0,0,0,0)=\mathbf{0}\end{aligned}$$

Also, using the commutative property (A5) (proved in Worked Exercise C22(a)) we have

$$\mathbf{v}+(-\mathbf{v})=\mathbf{0}=-\mathbf{v}+\mathbf{v},$$

so the additive inverses property (A4) holds.

Solution to Exercise C46

(a) $(p_1(x)+p_2(x))+p_3(x)$

$$\begin{aligned}&= ((a_1+b_1x+c_1x^2)+(a_2+b_2x+c_2x^2))\\ &\quad +(a_3+b_3x+c_3x^2)\\ &= ((a_1+a_2)+(b_1+b_2)x+(c_1+c_2)x^2)\\ &\quad +(a_3+b_3x+c_3x^2)\\ &= (a_1+a_2+a_3)+(b_1+b_2+b_3)x\\ &\quad +(c_1+c_2+c_3)x^2\end{aligned}$$

and

$p_1(x)+(p_2(x)+p_3(x))$

$$\begin{aligned}&= (a_1+b_1x+c_1x^2)\\ &\quad +((a_2+b_2x+c_2x^2)+(a_3+b_3x+c_3x^2))\\ &= (a_1+b_1x+c_1x^2)\\ &\quad +((a_2+a_3)+(b_2+b_3)x+(c_2+c_3)x^2)\\ &= (a_1+a_2+a_3)+(b_1+b_2+b_3)x\\ &\quad +(c_1+c_2+c_3)x^2.\end{aligned}$$

Therefore

$$\begin{aligned}&(p_1(x)+p_2(x))+p_3(x)\\ &\quad = p_1(x)+(p_2(x)+p_3(x)),\end{aligned}$$

and so the associative property (A2) holds for addition in P_3.

(b) We have $\mathbf{0}=0+0x+0x^2$, so

$$\begin{aligned}p_1(x)+\mathbf{0} &= (a_1+b_1x+c_1x^2)+(0+0x+0x^2)\\ &= (a_1+0)+(b_1+0)x+(c_1+0)x^2\\ &= a_1+b_1x+c_1x^2=p_1(x)\end{aligned}$$

Also, using the commutative property (A5) (proved in Worked Exercise C23(a)) we have

$$p_1(x)+\mathbf{0}=p_1(x)=\mathbf{0}+p_1(x),$$

so the additive identity property (A3) holds for addition in P_3.

Solution to Exercise C47

(a)
$$\begin{aligned}1\times p(x) &= 1\times(1-x+2x^2)\\ &= 1\times 1-1\times x+1\times 2x^2\\ &= 1-x+2x^2=p(x),\end{aligned}$$

and therefore the identity property (S3) holds here.

(b) $$\begin{aligned}\alpha(\beta p(x)) &= 2(-3(1-x+2x^2))\\ &= 2(-3+3x-6x^2)\\ &= -6+6x-12x^2\\ &= -6(1-x+2x^2) = (\alpha\beta)p(x),\end{aligned}$$

and therefore the associative property (S2) holds here.

Solution to Exercise C48

(a) Consider $(1,3)$ and $(2,5)$, both in V. Then $(1,3)+(2,5)=(3,8)$, which does not belong to the set V, since $2\times 3+1=7\neq 8$. So the set is not closed under vector addition.

Therefore the set of all ordered pairs (x,y) with $y=2x+1$ fails to satisfy the closure axiom (A1), so is not a real vector space.

Alternatively, note that for $(0,0)\in\mathbb{R}^2$ we have $2\times 0+1=1\neq 0$, so the zero vector is not in V and the additive identity axiom (A3) fails.

Other axioms also fail or do not make sense.

(b) Consider the matrix $\mathbf{A}=\begin{pmatrix}0 & 2\\ -1 & 3\end{pmatrix}$ and $\alpha=\frac{1}{2}$. Then $\alpha\mathbf{A}=\begin{pmatrix}0 & 1\\ -\frac{1}{2} & \frac{3}{2}\end{pmatrix}$, which does not belong to the set.

Therefore the set of matrices of the form

$$\begin{pmatrix}0 & a\\ b & c\end{pmatrix}\quad \text{with } a,b,c\in\mathbb{Z}$$

fails to satisfy the closure axiom (S1), so is not a real vector space.

Note that axioms A1–A5 and S3 do all hold here, but since axiom S1 fails, the axioms S2, D1 and D2 are meaningless.

Solution to Exercise C49

(a) $$\begin{aligned}4\mathbf{v}_1-2\mathbf{v}_2 &= 4(0,3)-2(2,1)\\ &= (0,12)-(4,2)=(-4,10)\end{aligned}$$

(b) $$\begin{aligned}3\mathbf{v}_1+2\mathbf{v}_2 &= 3(1,2,1,3)+2(2,1,0,-1)\\ &= (3,6,3,9)+(4,2,0,-2)\\ &= (7,8,3,7)\end{aligned}$$

Solution to Exercise C50

(a) $$\begin{aligned}2\mathbf{v}_1-4\mathbf{v}_2 &= 2(2-x+3x^2)-4(-1+x)\\ &= (4-2x+6x^2)-(-4+4x)\\ &= 8-6x+6x^2\end{aligned}$$

(b) $2\mathbf{v}_1-4\mathbf{v}_2 = 2\sin x-4x\cos x$

(c) $$\begin{aligned}2\mathbf{v}_1-4\mathbf{v}_2 &= 2\begin{pmatrix}-1 & 1\\ 2 & 0\end{pmatrix}-4\begin{pmatrix}3 & 1\\ 0 & -2\end{pmatrix}\\ &= \begin{pmatrix}-2 & 2\\ 4 & 0\end{pmatrix}-\begin{pmatrix}12 & 4\\ 0 & -8\end{pmatrix}\\ &= \begin{pmatrix}-14 & -2\\ 4 & 8\end{pmatrix}\end{aligned}$$

Solution to Exercise C51

We apply Strategy C6.

(a) Let α and β be real numbers such that

$$(2,4)=\alpha(0,3)+\beta(2,1)=(2\beta,3\alpha+\beta).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned}2\beta &= 2\\ 3\alpha+\ \beta &= 4.\end{aligned}$$

The first equation gives $\beta=1$, and substituting this into the second equation gives $\alpha=1$, so

$$(2,4)=(0,3)+(2,1).$$

(You might have spotted this linear combination without performing the calculations – it is always worth checking there is not an obvious solution before diving into a strategy!)

(b) Let α, β and γ be real numbers such that

$$\begin{aligned}(2,3,-2) &= \alpha(0,1,0)+\beta(1,2,-1)+\gamma(1,1,-2)\\ &= (\beta+\gamma,\alpha+2\beta+\gamma,-\beta-2\gamma).\end{aligned}$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned}\beta+\ \gamma &= 2\\ \alpha+\ 2\beta+\ \gamma &= 3\\ -\beta-2\gamma &= -2.\end{aligned}$$

Adding the first and third equations gives $\gamma=0$, and substituting this into the first equation gives $\beta=2$. Substituting both these values into the second equation gives $\alpha=-1$, so

$$(2,3,-2)=-(0,1,0)+2(1,2,-1)+0(1,1,-2).$$

(c) Let α and β be real numbers such that

$$\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix} = \alpha \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} + \beta \begin{pmatrix} 0 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & -\alpha - 2\beta \\ 0 & 2\alpha + \beta \end{pmatrix}.$$

Equating corresponding entries, we obtain the system

$$\begin{aligned} \alpha \phantom{{}-2\beta} &= 3 \\ -\alpha - 2\beta &= 1 \\ 2\alpha + \beta &= 4. \end{aligned}$$

The first equation gives $\alpha = 3$, and substituting this into the second equation gives $\beta = -2$. These values also satisfy the third equation, so

$$\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix} = 3 \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} - 2 \begin{pmatrix} 0 & -2 \\ 0 & 1 \end{pmatrix}.$$

Solution to Exercise C52

(a) We write

$$\begin{aligned} (1,5,4) &= \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 \\ &= \alpha(1,0,3) + \beta(0,2,0) = (\alpha, 2\beta, 3\alpha). \end{aligned}$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha \phantom{{}+2\beta} &= 1 \\ 2\beta &= 5 \\ 3\alpha \phantom{{}+2\beta} &= 4. \end{aligned}$$

This system is inconsistent and therefore has no solution. So $(1,5,4)$ does not lie in the subset of $\mathbb{R}^3$ spanned by $\{\mathbf{v}_1, \mathbf{v}_2\}$; that is, $(1,5,4)$ does not belong to $\langle\{\mathbf{v}_1, \mathbf{v}_2\}\rangle$.

(b) We write

$$\begin{aligned} (1,5,4) &= \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 + \gamma \mathbf{v}_3 \\ &= \alpha(1,0,3) + \beta(0,2,0) + \gamma(0,3,1) \\ &= (\alpha, 2\beta + 3\gamma, 3\alpha + \gamma). \end{aligned}$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha \phantom{{}+2\beta+3\gamma} &= 1 \\ 2\beta + 3\gamma &= 5 \\ 3\alpha \phantom{{}+2\beta} + \gamma &= 4. \end{aligned}$$

The first equation gives $\alpha = 1$, and substituting this into the third gives $\gamma = 1$. Substituting this into the second equation gives $\beta = 1$, so $(1,5,4)$ lies in the subset of $\mathbb{R}^3$ spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$; that is, $(1,5,4)$ belongs to $\langle\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}\rangle$ and it can be written as

$$(1,5,4) = 1(1,0,3) + 1(0,2,0) + 1(0,3,1).$$

(You might have spotted this and avoided following the formal method.)

Solution to Exercise C53

(a) Each vector in $\mathbb{R}^2$ can be written as (x,y). To show that (x,y) is in $\langle\{(1,1),(-1,2)\}\rangle$, we write

$$\begin{aligned} (x,y) &= \alpha(1,1) + \beta(-1,2) \\ &= (\alpha - \beta, \alpha + 2\beta). \end{aligned}$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha - \beta &= x \\ \alpha + 2\beta &= y. \end{aligned}$$

These equations have solution $\alpha = \frac{1}{3}(2x+y)$ and $\beta = \frac{1}{3}(y-x)$, so any vector in $\mathbb{R}^2$ can be written in terms of $(1,1)$ and $(-1,2)$ as

$$(x,y) = \tfrac{1}{3}(2x+y)(1,1) + \tfrac{1}{3}(y-x)(-1,2).$$

So $\{(1,1),(-1,2)\}$ is a spanning set for $\mathbb{R}^2$.

(b) Each vector in $\mathbb{R}^2$ can be written as (x,y). To show that (x,y) is in $\langle\{(2,-1),(3,2)\}\rangle$, we write

$$\begin{aligned} (x,y) &= \alpha(2,-1) + \beta(3,2) \\ &= (2\alpha + 3\beta, -\alpha + 2\beta). \end{aligned}$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} 2\alpha + 3\beta &= x \\ -\alpha + 2\beta &= y. \end{aligned}$$

These equations have solution $\alpha = \frac{1}{7}(2x-3y)$ and $\beta = \frac{1}{7}(x+2y)$, so any vector in $\mathbb{R}^2$ can be written in terms of $(2,-1)$ and $(3,2)$ as

$$(x,y) = \tfrac{1}{7}(2x-3y)(2,-1) + \tfrac{1}{7}(x+2y)(3,2).$$

So $\{(2,-1),(3,2)\}$ is a spanning set for $\mathbb{R}^2$.

Solution to Exercise C54

We write

$$\begin{aligned} (x,y,z) &= \alpha(1,0,0) + \beta(1,1,0) + \gamma(2,0,1) \\ &= (\alpha + \beta + 2\gamma, \beta, \gamma). \end{aligned}$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha + \beta + 2\gamma &= x \\ \beta \qquad &= y \\ \gamma &= z. \end{aligned}$$

Working backwards from the third equation, we find that these equations have solution $\gamma = z$, $\beta = y$ and $\alpha = x - y - 2z$, so any vector in $\mathbb{R}^3$ can be written in terms of $(1,0,0)$, $(1,1,0)$ and $(2,0,1)$ as

$$\begin{aligned} (x,y,z) &= (x-y-2z)(1,0,0) \\ &\quad + y(1,1,0) + z(2,0,1). \end{aligned}$$

So $\{(1,0,0),(1,1,0),(2,0,1)\}$ is a spanning set for $\mathbb{R}^3$.

Solution to Exercise C55

Each polynomial in P_4 can be written as $a + bx + cx^2 + dx^3$. To show that $a + bx + cx^2 + dx^3$ belongs to $\langle\{1+x, 1+x^2, 1+x^3, x\}\rangle$, we write

$$\begin{aligned} &a + bx + cx^2 + dx^3 \\ &\qquad = \alpha(1+x) + \beta(1+x^2) + \gamma(1+x^3) + \delta x \\ &\qquad = (\alpha+\beta+\gamma) + (\alpha+\delta)x + \beta x^2 + \gamma x^3. \end{aligned}$$

Equating corresponding coefficients, we obtain the system

$$\begin{aligned} \alpha + \beta + \gamma \qquad &= a \\ \alpha \qquad\qquad + \delta &= b \\ \beta \qquad\qquad &= c \\ \gamma \qquad &= d. \end{aligned}$$

It has solution $\gamma = d$, $\beta = c$, $\alpha = a - c - d$ and $\delta = b - a + c + d$. So

$$\begin{aligned} &a + bx + cx^2 + dx^3 \\ &\quad = (a-c-d)(1+x) + c(1+x^2) + d(1+x^3) \\ &\qquad + (b-a+c+d)x. \end{aligned}$$

Thus $\langle\{1+x, 1+x^2, 1+x^3, x\}\rangle = P_4$.

Solution to Exercise C56

(a) We have

$$\begin{aligned} \langle S\rangle &= \{\alpha(1,0,0) : \alpha \in \mathbb{R}\} \\ &= \{(\alpha,0,0) : \alpha \in \mathbb{R}\}. \end{aligned}$$

(Geometrically, $\langle S\rangle$ is the x-axis.)

(b) We have

$$\begin{aligned} \langle S\rangle &= \left\{\alpha\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} + \beta\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} : \alpha,\beta \in \mathbb{R}\right\} \\ &= \left\{\begin{pmatrix} 2\alpha-\beta & 0 \\ 0 & 3\alpha+2\beta \end{pmatrix} : \alpha,\beta \in \mathbb{R}\right\}. \end{aligned}$$

Thus

$$\langle S\rangle \subseteq \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a,b \in \mathbb{R}\right\}.$$

To show that every 2×2 diagonal matrix belongs to $\langle S\rangle$, we write

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 2\alpha-\beta & 0 \\ 0 & 3\alpha+2\beta \end{pmatrix}.$$

Equating corresponding entries, we obtain the system

$$\begin{aligned} 2\alpha - \beta &= a \\ 3\alpha + 2\beta &= b. \end{aligned}$$

It has solution

$$\begin{aligned} \alpha &= \tfrac{1}{7}(2a+b) \\ \beta &= \tfrac{1}{7}(-3a+2b), \end{aligned}$$

so

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \langle S\rangle.$$

Hence

$$\langle S\rangle = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a,b \in \mathbb{R}\right\}.$$

Solution to Exercise C57

(a) These two vectors are linearly independent because neither is a multiple of the other. (In this case there is no need to use Strategy C7.)

(b) Using Strategy C7, we write

$$\alpha(1,-1) + \beta(1,1) + \gamma(2,1) = (0,0).$$

This gives the system

$$\begin{aligned} \alpha + \beta + 2\gamma &= 0 \\ -\alpha + \beta + \gamma &= 0. \end{aligned}$$

Adding the equations gives $2\beta + 3\gamma = 0$, or $\beta = -\frac{3}{2}\gamma$, and substituting this into the first equation gives $\alpha = -\frac{1}{2}\gamma$; that is, $\gamma = -2\alpha$ and $\beta = 3\alpha$. The solution set of the system is

$$\alpha = k,\ \beta = 3k,\ \gamma = -2k, \quad k \in \mathbb{R},$$

so there are infinitely many solutions. For example, $k = 1$ gives

$$(1,-1) + 3(1,1) - 2(2,1) = (0,0).$$

So the set $\{(1,-1),(1,1),(2,1)\}$ is linearly dependent.

Alternatively, you may have expressed the solution set here in terms of γ and found another solution – *any* solution (where α, β and γ are not all zero) is sufficient to show that the vectors are linearly dependent.

(c) These two vectors are linearly independent because neither is a multiple of the other. (In this case there is no need to use Strategy C7.)

(d) We write

$$\alpha(1,0,0) + \beta(1,1,0) + \gamma(1,1,1) = (0,0,0).$$

This gives the system

$$\begin{aligned} \alpha + \beta + \gamma &= 0 \\ \beta + \gamma &= 0 \\ \gamma &= 0. \end{aligned}$$

The third equation gives $\gamma = 0$, and substituting into the second equation gives $\beta = 0$. Finally, substituting into the first equation gives $\alpha = 0$. The only solution is $\alpha = \beta = \gamma = 0$.

Therefore the set $\{(1,0,0),(1,1,0),(1,1,1)\}$ is linearly independent.

(e) These two vectors are linearly independent because neither is a multiple of the other. (Again, there is no need to use Strategy C7.)

Solution to Exercise C58

(a) The set $\{1, x, x^2, x^3, 1+x+x^2+x^3\}$ is linearly dependent because the fifth vector is the sum of the first four vectors. So

$$1 + x + x^2 + x^3 - (1 + x + x^2 + x^3) = 0.$$

(b) The set S is linearly independent because neither matrix is a multiple of the other.

(c) We apply Strategy C7.

We write

$$\alpha\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} + \beta\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} + \gamma\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},$$

which can be written as

$$\begin{pmatrix}\alpha+\beta+\gamma & \alpha+\gamma\\ \beta+\gamma & \alpha+\beta+\gamma\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

Equating corresponding entries, we obtain the system

$$\begin{aligned} \alpha + \beta + \gamma &= 0 \\ \alpha \qquad + \gamma &= 0 \\ \beta + \gamma &= 0 \\ \alpha + \beta + \gamma &= 0. \end{aligned}$$

Subtracting the second equation from the first, and the third from the fourth, we get $\beta = 0$ and $\alpha = 0$. Substituting these values in the first and fourth gives $\gamma = 0$ also. Therefore the only solution to this system is $\alpha = \beta = \gamma = 0$. Therefore the set S is a linearly independent subset of $M_{2,2}$.

(d) The set $\{1+i, 1-i\}$ is linearly independent because neither vector is a (real) multiple of the other.

Solution to Exercise C59

(a) None of the vectors in this set has a non-zero x-component; so whenever $x \neq 0$, we cannot write (x,y,z) in terms of these three vectors.

Therefore this set of vectors is not a basis for $\mathbb{R}^3$ because it does not span $\mathbb{R}^3$.

(If you had not spotted the zero x-component and had followed Strategy C8, you would have discovered that this set is not linearly independent: for example,

$$16(0,1,2) - 11(0,2,3) + (0,6,1) = (0,0,0).$$

Therefore this set of vectors is not a basis for $\mathbb{R}^3$.)

(b) We check both conditions in Strategy C8.

Using Strategy C7, we write

$$\alpha(1,2,1) + \beta(1,0,-1) + \gamma(0,3,1) = (0,0,0),$$

which simplifies to

$$(\alpha+\beta, 2\alpha+3\gamma, \alpha-\beta+\gamma) = (0,0,0).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha + \beta \qquad &= 0 \\ 2\alpha \qquad + 3\gamma &= 0 \\ \alpha - \beta + \gamma &= 0. \end{aligned}$$

Adding the third equation to the first gives $2\alpha + \gamma = 0$, and subtracting this from the second equation gives $\gamma = 0$. Substituting this into the second equation gives $\alpha = 0$. Finally, substituting $\alpha = 0$ into the first equation gives $\beta = 0$. The only solution is $\alpha = \beta = \gamma = 0$.

Therefore the set is linearly independent.

We apply Strategy C6.

Each vector in $\mathbb{R}^3$ can be written as (x, y, z), with $x, y, z \in \mathbb{R}$. To show that (x, y, z) is in

$$\langle\{(1,2,1),(1,0,-1),(0,3,1)\}\rangle,$$

we write

$$(x, y, z) = \alpha(1,2,1) + \beta(1,0,-1) + \gamma(0,3,1).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha + \beta \phantom{{}+3\gamma} &= x \\ 2\alpha \phantom{{}+\beta} + 3\gamma &= y \\ \alpha - \beta + \gamma &= z. \end{aligned}$$

Adding the third equation to the first gives $2\alpha + \gamma = x + z$, and subtracting this from the second equation gives $\gamma = \frac{1}{2}(y - x - z)$. Substituting this into the second equation gives $\alpha = \frac{1}{4}(3x - y + 3z)$. Finally, substituting for α in the first equation gives $\beta = \frac{1}{4}(x + y - 3z)$. We have a solution, so any vector in $\mathbb{R}^3$ can be written as

$$\begin{aligned}(x, y, z) = {}& \tfrac{1}{4}(3x - y + 3z)(1,2,1) \\ &+ \tfrac{1}{4}(x + y - 3z)(1,0,-1) \\ &+ \tfrac{1}{2}(y - x - z)(0,3,1).\end{aligned}$$

Therefore the set of vectors spans $\mathbb{R}^3$.

Thus $\{(1,2,1),(1,0,-1),(0,3,1)\}$ is a basis for $\mathbb{R}^3$.

(c) Here we have

$$(1,1,1) = (1,0,0) + (0,1,0) + (0,0,1),$$

so these vectors are not linearly independent.

Therefore the set

$$\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$$

is not a basis for $\mathbb{R}^3$.

Solution to Exercise C60

We check both conditions in Strategy C8.

This set is linearly independent because there are only two vectors in the set, and neither vector is a multiple of the other.

We apply Strategy C6.

Each vector in $\mathbb{R}^4$ can be written as (x, y, z, w), with $x, y, z, w \in \mathbb{R}$. To show that (x, y, z, w) is in

$$\langle\{(1,2,-1,-1),(-1,5,1,3)\}\rangle,$$

we write

$$(x, y, z, w) = \alpha(1,2,-1,-1) + \beta(-1,5,1,3).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha - \beta &= x \\ 2\alpha + 5\beta &= y \\ -\alpha + \beta &= z \\ -\alpha + 3\beta &= w. \end{aligned}$$

Adding the first and third equations gives $x + z = 0$. This contradicts the assumption that x, y, z and w can take any real values, so

$$\{(1,2,-1,-1),(-1,5,1,3)\}$$

is not a spanning set for $\mathbb{R}^4$.

Thus the set $\{(1,2,-1,-1),(-1,5,1,3)\}$ is not a basis for $\mathbb{R}^4$.

Solution to Exercise C61

We check both conditions in Strategy C8.

Using Strategy C7 we write

$$\alpha\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} + \beta\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} + \gamma\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix} + \delta\begin{pmatrix}-3 & 1\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

which simplifies to

$$\begin{pmatrix}\alpha + 2\gamma - 3\delta & -\beta + \delta\\ \alpha + \beta & \gamma\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

Equating corresponding entries, we obtain the system

$$\begin{aligned} \alpha \phantom{{}+\beta} + 2\gamma - 3\delta &= 0 \\ -\beta \phantom{{}+2\gamma} + \delta &= 0 \\ \alpha + \beta \phantom{{}+2\gamma-3\delta} &= 0 \\ \gamma \phantom{{}-3\delta} &= 0. \end{aligned}$$

From the fourth equation we have $\gamma = 0$, and from the second and third $\alpha = -\beta = -\delta$. Substituting into the first equation gives $\alpha + 3\alpha = 0$, so $\alpha = 0$. The only solution is therefore $\alpha = \beta = \gamma = \delta = 0$.

Therefore the set is linearly independent.

We apply Strategy C6.

Each 2×2 matrix can be written as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $a, b, c, d \in \mathbb{R}$. To show this is in $\langle S \rangle$ we write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \delta \begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix}.$$

Equating corresponding entries, we obtain the system

$$\begin{aligned} \alpha \phantom{{}+\beta} &+ 2\gamma - 3\delta = a \\ -\beta &\phantom{{}+2\gamma} + \delta = b \\ \alpha + \beta &\phantom{{}+2\gamma-3\delta} = c \\ \gamma &\phantom{{}-3\delta} = d. \end{aligned}$$

From the fourth equation we have $\gamma = d$, and adding the second equation to the third gives $\alpha + \delta = b + c$. Substituting for γ in the first equation gives $\alpha - 3\delta = a - 2d$. These last two equations give $\delta = \frac{1}{4}(b + c - a + 2d)$.

Then, by substitution, $\alpha = \frac{1}{4}(a + 3b + 3c - 2d)$ and $\beta = \frac{1}{4}(-a - 3b + c + 2d)$.

We have a solution $\alpha = \frac{1}{4}(a + 3b + 3c - 2d)$, $\beta = \frac{1}{4}(-a - 3b + c + 2d)$, $\gamma = d$ and $\delta = \frac{1}{4}(b + c - a + 2d)$.

Therefore the set of matrices S spans the set $M_{2,2}$ of all 2×2 matrices.

Thus S is a basis for $M_{2,2}$.

Solution to Exercise C62

(a) For the basis $E = \{(1,2), (-3,1)\}$, we have

$$\begin{aligned} (2,1)_E &= 2(1,2) + 1(-3,1) \\ &= (2,4) + (-3,1) \\ &= (-1,5). \end{aligned}$$

(b) For the basis $E = \{(1,0,2), (-1,1,3), (2,-2,0)\}$, we have

$$\begin{aligned} (1,1,-1)_E &= 1(1,0,2) + 1(-1,1,3) - 1(2,-2,0) \\ &= (1,0,2) + (-1,1,3) - (2,-2,0) \\ &= (-2,3,5). \end{aligned}$$

Solution to Exercise C63

(a) We write

$$(5,-4) = \alpha(1,2) + \beta(-3,1).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha - 3\beta &= 5 \\ 2\alpha + \beta &= -4. \end{aligned}$$

Solving these equations gives $\alpha = -1$, $\beta = -2$, so

$$\begin{aligned} (5,-4) &= -1(1,2) - 2(-3,1) \\ &= (-1,-2)_E. \end{aligned}$$

(b) We write

$$(-3,5,7) = \alpha(1,0,2) + \beta(-1,1,3) + \gamma(2,-2,0).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha - \beta + 2\gamma &= -3 \\ \beta - 2\gamma &= 5 \\ 2\alpha + 3\beta \phantom{{}+2\gamma} &= 7. \end{aligned}$$

Adding the first and second equations gives $\alpha = 2$, and substituting this into the third equation gives $\beta = 1$. Substituting for β in the second equation gives $\gamma = -2$. So

$$\begin{aligned} (-3,5,7) &= 2(1,0,2) + 1(-1,1,3) - 2(2,-2,0) \\ &= (2,1,-2)_E. \end{aligned}$$

Solution to Exercise C64

We apply Strategy C9.

(a) This set contains only two vectors, not three, so cannot be a basis for $\mathbb{R}^3$.

(Neither vector is a multiple of the other, so it is however linearly independent.)

(b) This set contains three vectors, so it may be a basis for $\mathbb{R}^3$.

We write

$$\alpha(1,0,1) + \beta(1,0,-1) + \gamma(0,1,1) = (0,0,0).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha + \beta \qquad &= 0 \\ \gamma &= 0 \\ \alpha - \beta + \gamma &= 0. \end{aligned}$$

The second equation gives $\gamma = 0$. Substituting this into the third equation gives $\alpha - \beta = 0$. Adding this new equation to the first equation gives $\alpha = 0$ and hence $\beta = 0$. The only solution is $\alpha = \beta = \gamma = 0$.

Therefore the set is linearly independent.

The set contains three vectors and is linearly independent; therefore it is a basis for $\mathbb{R}^3$.

(c) Here we have

$$(1, -1, 0) + (2, 1, 4) = (3, 0, 4),$$

so this set is not linearly independent.

Therefore this set is not a basis for $\mathbb{R}^3$.

(It does however contain the correct number of vectors.)

(d) This set contains four vectors, so it cannot be a basis for $\mathbb{R}^3$.

(Alternatively, here we have

$$(1, 1, 1) = (1, 0, 0) + (0, 1, 0) + (0, 0, 1),$$

so this set is also linearly dependent.)

Solution to Exercise C65

We apply Strategy C9.

(a) This set contains four vectors and $M_{2,2}$ has dimension 4, so it may be a basis.

Using Strategy C7 we write

$$\alpha \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + \gamma \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \delta \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which simplifies to

$$\begin{pmatrix} \alpha + \gamma & \beta + \gamma + \delta \\ \alpha + \delta & \beta + \delta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Equating corresponding entries, we obtain the system

$$\begin{aligned} \alpha \quad + \gamma \qquad &= 0 \\ \beta + \gamma + \delta &= 0 \\ \alpha \qquad\quad + \delta &= 0 \\ \beta \qquad + \delta &= 0. \end{aligned}$$

From the first, third and fourth equations we have $\alpha = \beta = -\gamma = -\delta$. Substituting in the second gives $-\beta = 0$. The only solution is therefore $\alpha = \beta = \gamma = \delta = 0$.

Therefore the set is linearly independent.

The set S contains four vectors and is linearly independent so is a basis for $M_{2,2}$.

(Compare the length of this solution to that of Exercise C61 using Strategy C8.)

(b) This set contains two vectors and P_2 has dimension 2, so it may be a basis.

This set is linearly independent because there are only two vectors in the set, and neither vector is a multiple of the other.

So by Strategy C9, the set is a basis for P_2.

Solution to Exercise C66

The set S is a subset of $\mathbb{R}^2$, so we use Strategy C10.

If $x = 0$, then $(x, -2x) = (0, 0)$, so S contains the zero vector of $\mathbb{R}^2$.

Let $\mathbf{v}_1 = (x_1, -2x_1)$ and $\mathbf{v}_2 = (x_2, -2x_2)$ belong to S. Then

$$\begin{aligned} \mathbf{v}_1 + \mathbf{v}_2 &= (x_1, -2x_1) + (x_2, -2x_2) \\ &= (x_1 + x_2, -2x_1 - 2x_2) \\ &= (x_1 + x_2, -2(x_1 + x_2)). \end{aligned}$$

This vector has the correct form for a vector in S, since $x_1 + x_2 \in \mathbb{R}$, so S is closed under vector addition.

Let $\mathbf{v} = (x, -2x) \in S$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} \alpha\mathbf{v} &= \alpha(x, -2x) \\ &= (\alpha x, \alpha(-2x)) \\ &= (\alpha x, -2(\alpha x)). \end{aligned}$$

This vector has the correct form for a vector in S, since $\alpha x \in \mathbb{R}$, so S is closed under scalar multiplication.

Since conditions (1), (2) and (3) are satisfied, S is a subspace of $\mathbb{R}^2$.

(This subspace is the line through the origin with equation $y = -2x$.)

Solution to Exercise C67

In each case the set S is a subset of V, so we use Strategy C10.

(a) If $\mathbf{0} \in S$, then $(x, x+2) = (0,0)$ for some number x. Equating coordinates, we obtain the system

$$\begin{aligned} x &= 0 \\ x &= -2. \end{aligned}$$

This system is inconsistent so has no solution. Therefore $\mathbf{0}$ does not belong to S and condition (1) is not satisfied. Hence S is not a subspace of $\mathbb{R}^2$.

(b) If $x = y = z = 0$, then

$$(x, y, z, x + 2y - z) = (0,0,0,0),$$

so S contains the zero vector of $\mathbb{R}^4$.

Let $\mathbf{v}_1 = (x_1, y_1, z_1, x_1 + 2y_1 - z_1)$ and $\mathbf{v}_2 = (x_2, y_2, z_2, x_2 + 2y_2 - z_2)$ belong to S. Then

$$\begin{aligned} \mathbf{v}_1 + \mathbf{v}_2 &= (x_1, y_1, z_1, x_1 + 2y_1 - z_1) \\ &\quad + (x_2, y_2, z_2, x_2 + 2y_2 - z_2) \\ &= (x_1 + x_2, y_1 + y_2, z_1 + z_2, \\ &\qquad x_1 + 2y_1 - z_1 + x_2 + 2y_2 - z_2) \\ &= (x_1 + x_2, y_1 + y_2, z_1 + z_2, \\ &\qquad (x_1 + x_2) + 2(y_1 + y_2) - (z_1 + z_2)). \end{aligned}$$

This vector has the correct form for a vector in S, since $x_1 + x_2, y_1 + y_2, z_1 + z_2 \in \mathbb{R}$, so S is closed under vector addition.

Let $\mathbf{v} = (x, y, z, x + 2y - z) \in S$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} \alpha\mathbf{v} &= \alpha(x, y, z, x + 2y - z) \\ &= (\alpha x, \alpha y, \alpha z, \alpha(x + 2y - z)) \\ &= (\alpha x, \alpha y, \alpha z, (\alpha x) + 2(\alpha y) - (\alpha z)). \end{aligned}$$

This vector has the correct form for a vector in S, since $\alpha x, \alpha y, \alpha z \in \mathbb{R}$, so S is closed under scalar multiplication.

Since conditions (1), (2) and (3) are satisfied, S is a subspace of $\mathbb{R}^4$.

Solution to Exercise C68

In each case the set S is a subset of V, so we use Strategy C10.

(a) The zero vector of P_3 is $0 + 0x + 0x^2 = \mathbf{0}$. If $a = b = 0$, then $p(x) = 0 + 0x = \mathbf{0}$, so S contains the zero vector.

Let $p_1(x) = a_1 + b_1x$ and $p_2(x) = a_2 + b_2x$ belong to S. Then

$$\begin{aligned} p_1(x) + p_2(x) &= a_1 + b_1x + a_2 + b_2x \\ &= (a_1 + a_2) + (b_1 + b_2)x. \end{aligned}$$

This polynomial has the correct form for a vector in S, since $a_1 + a_2, b_1 + b_2 \in \mathbb{R}$, so S is closed under vector addition.

Let $p(x) = a + bx \in S$ and $\alpha \in \mathbb{R}$. Then

$$\alpha p(x) = \alpha a + \alpha bx = (\alpha a) + (\alpha b)x.$$

This polynomial has the correct form for a vector in S, since $\alpha a, \alpha b \in \mathbb{R}$, so S is closed under scalar multiplication.

Since conditions (1), (2) and (3) are satisfied, S is a subspace of V.

(b) The zero vector of P_3 is $0 + 0x + 0x^2 = \mathbf{0}$, which is not of the form $x + ax^2$ for a vector in S. Therefore $\mathbf{0}$ does not belong to S and condition (1) fails. Hence S is not a subspace of P_3.

(Alternatively, you may have spotted that conditions (2) and (3) also fail. Using a particularly simple vector can make the calculations to show this easy: by setting for example $a = 0$, we see that $p(x) = x$ belongs to S. The sum $p(x) + p(x) = 2x$, however, does not belong to S, and for $\alpha \in \mathbb{R}$ not equal to 1, the scalar product αx is also not in S.)

(c) The zero vector of $M_{2,2}$ is $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, which is not of the form $\begin{pmatrix} a & 1 \\ 0 & d \end{pmatrix}$ for a vector in S.

Therefore $\mathbf{0}$ does not belong to S and condition (1) fails. Hence S is not a subspace of $M_{2,2}$.

Solution to Exercise C69

Since $\{(1,-2,0),(0,3,3)\}$ is a linearly independent set, the subspace it spans is a two-dimensional subspace of $\mathbb{R}^3$ and is therefore a plane through the origin with equation

$$ax + by + cz = 0,$$

where a, b, c are not all zero.

Since the vectors in the spanning set lie in the plane, the values of a, b and c must satisfy the system

$$\begin{aligned} a - 2b &= 0 \\ 3b + 3c &= 0. \end{aligned}$$

The first of these equations gives $a = 2b$, and the second equation gives $c = -b$, so the subspace is the plane with equation $2bx + by - bz = 0$, or, equivalently,

$$2x + y - z = 0.$$

Solution to Exercise C70

Since

$$\begin{aligned} &(x, y, z, x + 2y - z) \\ &\quad = (x, 0, 0, x) + (0, y, 0, 2y) + (0, 0, z, -z) \\ &\quad = x(1, 0, 0, 1) + y(0, 1, 0, 2) + z(0, 0, 1, -1), \end{aligned}$$

any vector in S can be written as a linear combination of the vectors in the set

$$\{(1, 0, 0, 1), (0, 1, 0, 2), (0, 0, 1, -1)\},$$

so this set spans S.

To check whether these vectors are linearly independent, we write

$$\begin{aligned} &\alpha(1, 0, 0, 1) + \beta(0, 1, 0, 2) + \gamma(0, 0, 1, -1) \\ &\quad = (0, 0, 0, 0). \end{aligned}$$

This gives the system

$$\begin{aligned} \alpha &= 0 \\ \beta &= 0 \\ \gamma &= 0 \\ \alpha + 2\beta - \gamma &= 0, \end{aligned}$$

and hence $\alpha = \beta = \gamma = 0$. Therefore the set is linearly independent.

So $\{(1, 0, 0, 1), (0, 1, 0, 2), (0, 0, 1, -1)\}$ is a basis for S. Therefore S has dimension 3.

Solution to Exercise C71

(a) $\begin{aligned}[t] (2, 1, 1) \cdot (1, -4, 2) &= 2 \times 1 + 1 \times (-4) + 1 \times 2 \\ &= 2 - 4 + 2 = 0, \end{aligned}$

so $(2, 1, 1)$ and $(1, -4, 2)$ are orthogonal.

(b) $\mathbf{v}_1 \cdot \mathbf{v}_2 = -2 \times 9 + 6 \times 2 + 1 \times 6 = 0,$

so $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_3 &= -2 \times 4 + 6 \times (-15) + 1 \times (-1) \\ &= -99, \end{aligned}$$

which is non-zero, so $\mathbf{v}_1$ and $\mathbf{v}_3$ are not orthogonal.

$$\begin{aligned} \mathbf{v}_2 \cdot \mathbf{v}_3 &= 9 \times 4 + 2 \times (-15) \\ &\qquad + 6 \times (-1) = 0, \end{aligned}$$

so $\mathbf{v}_2$ and $\mathbf{v}_3$ are orthogonal.

Solution to Exercise C72

(a) Let $\mathbf{v}_1 = (3, 4, 0)$, $\mathbf{v}_2 = (8, -6, 0)$ and $\mathbf{v}_3 = (0, 0, 5)$. Then

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_2 &= 3 \times 8 + 4 \times (-6) + 0 \times 0 = 0, \\ \mathbf{v}_1 \cdot \mathbf{v}_3 &= 3 \times 0 + 4 \times 0 + 0 \times 5 = 0, \\ \mathbf{v}_2 \cdot \mathbf{v}_3 &= 8 \times 0 + (-6) \times 0 + 0 \times 5 = 0. \end{aligned}$$

Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set in $\mathbb{R}^3$. Since there are three non-zero vectors in this set, it is an orthogonal basis for $\mathbb{R}^3$.

(b) We apply Strategy C11.

$$\begin{aligned} \alpha_1 &= \frac{\mathbf{v}_1 \cdot \mathbf{u}}{\mathbf{v}_1 \cdot \mathbf{v}_1} \\ &= \frac{(3, 4, 0) \cdot (10, 0, 4)}{(3, 4, 0) \cdot (3, 4, 0)} \\ &= \frac{30}{25} = \frac{6}{5}, \\ \alpha_2 &= \frac{\mathbf{v}_2 \cdot \mathbf{u}}{\mathbf{v}_2 \cdot \mathbf{v}_2} \\ &= \frac{(8, -6, 0) \cdot (10, 0, 4)}{(8, -6, 0) \cdot (8, -6, 0)} \\ &= \frac{80}{100} = \frac{4}{5}, \end{aligned}$$

and

$$\begin{aligned} \alpha_3 &= \frac{\mathbf{v}_3 \cdot \mathbf{u}}{\mathbf{v}_3 \cdot \mathbf{v}_3} \\ &= \frac{(0, 0, 5) \cdot (10, 0, 4)}{(0, 0, 5) \cdot (0, 0, 5)} \\ &= \frac{20}{25} = \frac{4}{5}. \end{aligned}$$

Thus $(10, 0, 4) = \frac{6}{5}(3, 4, 0) + \frac{4}{5}(8, -6, 0) + \frac{4}{5}(0, 0, 5)$.

Solution to Exercise C73

(a)
$$\begin{aligned}(1,2,-1,0)\cdot(0,-5,6,-3) &= 1\times 0+2\times(-5)+(-1)\times 6+0\times(-3)\\ &= 0-10-6+0\\ &= -16\end{aligned}$$

(b)
$$\begin{aligned}(1,2,3,4,5,6)\cdot(3,2,1,0,-1,-2) &= 1\times 3+2\times 2+3\times 1+4\times 0\\ &\quad +5\times(-1)+6\times(-2)\\ &= 3+4+3+0-5-12\\ &= -7\end{aligned}$$

Solution to Exercise C74

We check that each pair of vectors is orthogonal by forming the scalar product of each pair of vectors in the set:

$$\begin{aligned}(1,0,0,0,0)\cdot(0,2,0,0,0) &= 0+0+0+0+0\\ &= 0,\\ (1,0,0,0,0)\cdot(0,0,1,1,0) &= 0+0+0+0+0\\ &= 0,\\ (0,2,0,0,0)\cdot(0,0,1,1,0) &= 0+0+0+0+0\\ &= 0.\end{aligned}$$

Therefore these three vectors form an orthogonal set in $\mathbb{R}^5$.

Solution to Exercise C75

We check that each pair of vectors is orthogonal by forming the scalar product of each pair of vectors in the set:

$$\begin{aligned}(1,2,1,0)\cdot(-1,1,-1,1) &= -1+2-1+0\\ &= 0,\\ (1,2,1,0)\cdot(1,0,-1,0) &= 1+0-1+0\\ &= 0,\\ (1,2,1,0)\cdot(1,-1,1,3) &= 1-2+1+0\\ &= 0,\\ (-1,1,-1,1)\cdot(1,0,-1,0) &= -1+0+1+0\\ &= 0,\\ (-1,1,-1,1)\cdot(1,-1,1,3) &= -1-1-1+3\\ &= 0,\\ (1,0,-1,0)\cdot(1,-1,1,3) &= 1+0-1+0\\ &= 0.\end{aligned}$$

Therefore these vectors form an orthogonal set in $\mathbb{R}^4$. Since there are four, non-zero vectors in this set, these vectors form an orthogonal basis for $\mathbb{R}^4$ by Corollary C33.

Solution to Exercise C76

We apply Strategy C12.

Let $\mathbf{v}_1=(1,2,1,0)$, $\mathbf{v}_2=(-1,1,-1,1)$, $\mathbf{v}_3=(1,0,-1,0)$, $\mathbf{v}_4=(1,-1,1,3)$ and $\mathbf{u}=(1,2,3,4)$. Then

$$\alpha_1=\frac{\mathbf{v}_1\cdot\mathbf{u}}{\mathbf{v}_1\cdot\mathbf{v}_1}=\frac{8}{6}=\frac{4}{3},$$

$$\alpha_2=\frac{\mathbf{v}_2\cdot\mathbf{u}}{\mathbf{v}_2\cdot\mathbf{v}_2}=\frac{2}{4}=\frac{1}{2},$$

$$\alpha_3=\frac{\mathbf{v}_3\cdot\mathbf{u}}{\mathbf{v}_3\cdot\mathbf{v}_3}=\frac{-2}{2}=-1,$$

$$\alpha_4=\frac{\mathbf{v}_4\cdot\mathbf{u}}{\mathbf{v}_4\cdot\mathbf{v}_4}=\frac{14}{12}=\frac{7}{6}.$$

Thus

$$\begin{aligned}(1,2,3,4) &= \tfrac{4}{3}(1,2,1,0)+\tfrac{1}{2}(-1,1,-1,1)\\ &\quad -(1,0,-1,0)+\tfrac{7}{6}(1,-1,1,3).\end{aligned}$$

Solution to Exercise C77

(a) Using $\mathbf{x}\cdot\mathbf{n}=0$, we have

$$(x,y,z)\cdot(3,-4,5)=0;$$

that is, the equation of the plane is

$$3x-4y+5z=0.$$

(b) We have

$$\begin{aligned}\mathbf{w}_1\cdot\mathbf{n} &= (4,3,0)\cdot(3,-4,5)\\ &= 12-12+0=0,\end{aligned}$$

and

$$\begin{aligned}\mathbf{w}_2\cdot\mathbf{n} &= (0,5,4)\cdot(3,-4,5)\\ &= 0-20+20=0,\end{aligned}$$

so both these vectors lie in the plane.

(Alternatively, rather than using the vector equation of the plane, we can check that the points $(4,3,0)$ and $(0,5,4)$ satisfy the equation $3x-4y+5z=0$ of the plane.)

(c) We set $\mathbf{v}_1 = (4,3,0)$ and

$$\begin{aligned}\mathbf{v}_2 &= \mathbf{w}_2 - \frac{\mathbf{v}_1 \cdot \mathbf{w}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\ &= (0,5,4) - \frac{(4,3,0)\cdot(0,5,4)}{(4,3,0)\cdot(4,3,0)}(4,3,0) \\ &= (0,5,4) - \tfrac{15}{25}(4,3,0) \\ &= (0,5,4) - \tfrac{3}{5}(4,3,0) \\ &= \left(-\tfrac{12}{5}, \tfrac{16}{5}, 4\right).\end{aligned}$$

The required orthogonal basis for the plane is

$$\left\{(4,3,0), \left(-\tfrac{12}{5}, \tfrac{16}{5}, 4\right)\right\}.$$

(d) An orthogonal basis for $\mathbb{R}^3$ is

$$\left\{(3,-4,5), (4,3,0), \left(-\tfrac{12}{5}, \tfrac{16}{5}, 4\right)\right\}.$$

Solution to Exercise C78

We apply Theorem C35 with $\mathbf{w}_1 = (1,0,0,0,0)$, $\mathbf{w}_2 = (0,2,0,0,0)$, $\mathbf{w}_3 = (0,0,1,1,0)$, $\mathbf{w}_4 = (1,1,1,1,1)$ and $\mathbf{w}_5 = (1,0,-1,0,1)$.

Since $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ already form an orthogonal set, we have

$$\begin{aligned}\mathbf{v}_1 &= \mathbf{w}_1 = (1,0,0,0,0), \\ \mathbf{v}_2 &= \mathbf{w}_2 = (0,2,0,0,0), \\ \mathbf{v}_3 &= \mathbf{w}_3 = (0,0,1,1,0).\end{aligned}$$

Then

$$\begin{aligned}\mathbf{v}_4 &= \mathbf{w}_4 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_4}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_4}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\ &\qquad - \left(\frac{\mathbf{v}_3 \cdot \mathbf{w}_4}{\mathbf{v}_3 \cdot \mathbf{v}_3}\right)\mathbf{v}_3 \\ &= (1,1,1,1,1) - \tfrac{1}{1}(1,0,0,0,0) \\ &\qquad - \tfrac{2}{4}(0,2,0,0,0) - \tfrac{2}{2}(0,0,1,1,0) \\ &= (0,0,0,0,1)\end{aligned}$$

and

$$\begin{aligned}\mathbf{v}_5 &= \mathbf{w}_5 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_5}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_5}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\ &\qquad - \left(\frac{\mathbf{v}_3 \cdot \mathbf{w}_5}{\mathbf{v}_3 \cdot \mathbf{v}_3}\right)\mathbf{v}_3 - \left(\frac{\mathbf{v}_4 \cdot \mathbf{w}_5}{\mathbf{v}_4 \cdot \mathbf{v}_4}\right)\mathbf{v}_4 \\ &= (1,0,-1,0,1) - \tfrac{1}{1}(1,0,0,0,0) \\ &\qquad - 0 - \left(-\tfrac{1}{2}\right)(0,0,1,1,0) - \tfrac{1}{1}(0,0,0,0,1) \\ &= \left(0,0,-\tfrac{1}{2},\tfrac{1}{2},0\right).\end{aligned}$$

Thus we have the orthogonal basis

$$\left\{(1,0,0,0,0), (0,2,0,0,0), (0,0,1,1,0), (0,0,0,0,1), \left(0,0,-\tfrac{1}{2},\tfrac{1}{2},0\right)\right\}.$$

Solution to Exercise C79

(a) $$\begin{aligned}(3,-4,5)\cdot(3,-4,5) &= 9 + 16 + 25 \\ &= 50,\end{aligned}$$

so $|(3,-4,5)| = \sqrt{50} = 5\sqrt{2}$.

(b) $$\begin{aligned}(1,2,-1,0,3)&\cdot(1,2,-1,0,3) \\ &= 1 + 4 + 1 + 0 + 9 \\ &= 15,\end{aligned}$$

so $|(1,2,-1,0,3)| = \sqrt{15}$.

Solution to Exercise C80

If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a non-zero vector, then

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{v_1}{\sqrt{\mathbf{v}\cdot\mathbf{v}}}, \frac{v_2}{\sqrt{\mathbf{v}\cdot\mathbf{v}}}, \ldots, \frac{v_n}{\sqrt{\mathbf{v}\cdot\mathbf{v}}}\right),$$

so the magnitude of $\mathbf{v}/|\mathbf{v}|$ is

$$\begin{aligned}&\sqrt{\frac{v_1^2}{\mathbf{v}\cdot\mathbf{v}} + \frac{v_2^2}{\mathbf{v}\cdot\mathbf{v}} + \cdots + \frac{v_n^2}{\mathbf{v}\cdot\mathbf{v}}} \\ &\quad = \sqrt{\frac{v_1^2 + v_2^2 + \cdots + v_n^2}{\mathbf{v}\cdot\mathbf{v}}} \\ &\quad = \sqrt{\frac{\mathbf{v}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}} = 1.\end{aligned}$$

Solution to Exercise C81

We apply Strategy C13.

We have

$$\begin{aligned}&|(1,2,1,0)| = \sqrt{6}, \\ &|(-1,1,-1,1)| = \sqrt{4} = 2, \\ &|(1,0,-1,0)| = \sqrt{2}, \\ &|(1,-1,1,3)| = \sqrt{12} = 2\sqrt{3}.\end{aligned}$$

The required orthonormal basis for $\mathbb{R}^4$ is therefore

$$\left\{\frac{1}{\sqrt{6}}(1,2,1,0), \frac{1}{2}(-1,1,-1,1), \frac{1}{\sqrt{2}}(1,0,-1,0), \frac{1}{2\sqrt{3}}(1,-1,1,3)\right\}.$$

Unit C3

Linear transformations

Introduction

In this unit you will study functions between vector spaces. You will begin by looking more closely at some particular functions that have $\mathbb{R}^2$ as their domain and codomain, such as rotations and reflections. These functions map parallel lines to parallel lines, preserve scalar multiples and map the zero vector to itself. Algebraically, these functions preserve the operations of addition and scalar multiplication in the vector space $\mathbb{R}^2$. There is a special name for functions that preserve addition and scalar multiplication between vector spaces: they are called *linear transformations.* You will see that such functions have a matrix representation. This link between linear transformations and matrices enables us to relate the properties of matrices with those of linear transformations. Finally, you will meet an important result concerning linear transformations, known as the *Dimension Theorem.* This theorem has a number of consequences. For example, it enables us to show how the number of solutions of a system of m linear equations in n unknowns depends on the values of m and n.

Many results from Units C1 *Linear equations and matrices* and C2 *Vector spaces* are used in this unit; so make sure that you understand the main ideas of those units before starting your study of this one.

1 Introducing linear transformations

In this section you will see that we can generalise properties of functions that have $\mathbb{R}^2$ as their domain and codomain to functions between other vector spaces.

1.1 What is a linear transformation?

We begin by investigating the properties of some simple but important functions, often called transformations, which map the vector space $\mathbb{R}^2$ to itself. For each one, a diagram shows the effect of the transformation on the square whose corners are at $(0,0)$, $(0,1)$, $(1,1)$ and $(1,0)$, and the effect on the vector $(1,1)$; part of the square is shaded for clarity.

We will investigate the following four functions: dilation, scaling, rotation and reflection.

For any real number k, a k-**dilation** of $\mathbb{R}^2$ scales (or stretches) vectors by a factor k with respect to the origin.

When $k = 2$, the magnitude of a vector is doubled, as illustrated in Figure 1.

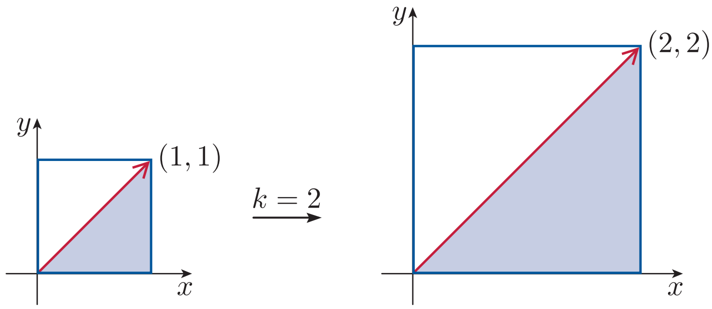


Figure 1 A 2-dilation

When $k = \frac{1}{2}$, the magnitude of a vector is halved, as illustrated in Figure 2.

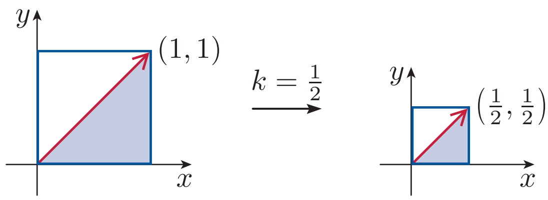


Figure 2 A $\frac{1}{2}$-dilation

When k is negative, the direction of a vector is reversed – as illustrated in Figure 3 for the case $k = -2$.

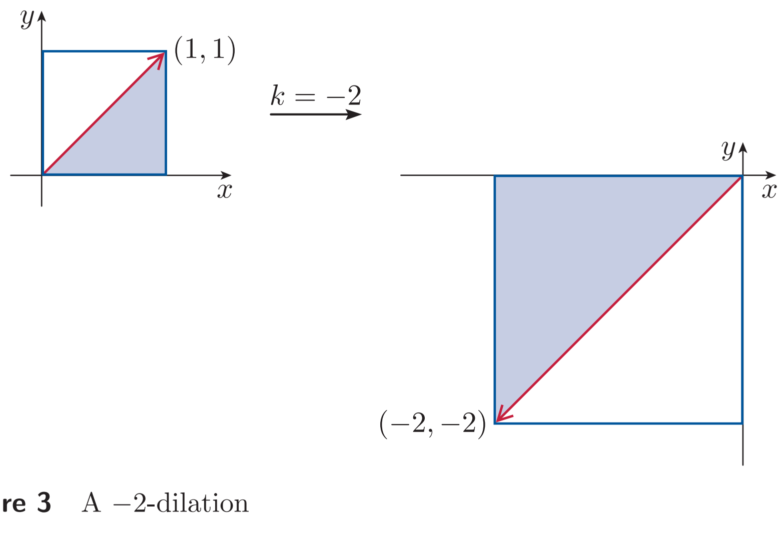


Figure 3 A -2-dilation

For any real numbers k and l, a **(k,l)-scaling** of $\mathbb{R}^2$ scales vectors by a factor k in the x-direction and by a factor l in the y-direction. Figure 4 shows the effect of a $\left(2,\frac{1}{2}\right)$-scaling.

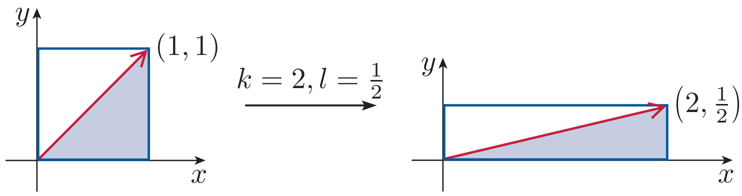


Figure 4 A $\left(2,\frac{1}{2}\right)$-scaling

Figure 5 shows the effect of a $(-1,3)$-scaling.

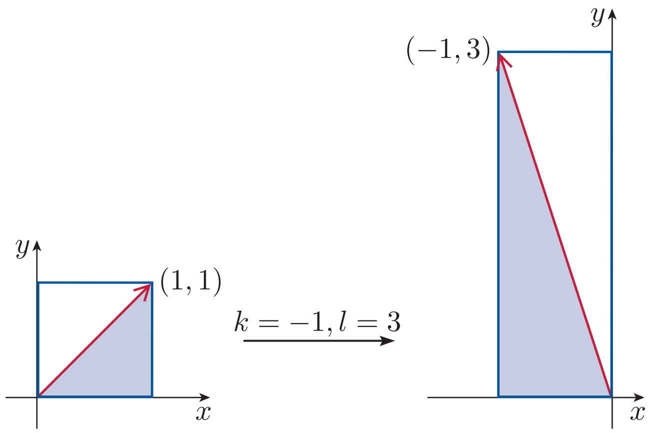


Figure 5 A $(-1,3)$-scaling

A **rotation** r_θ of $\mathbb{R}^2$ rotates vectors anticlockwise through an angle θ about the origin $(0,0)$.

Figure 6 shows the effect of a rotation $r_{\pi/4}$.

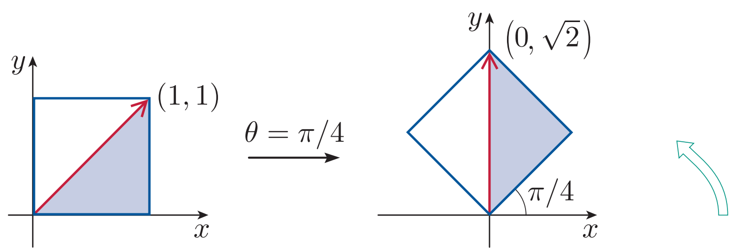


Figure 6 A rotation $r_{\pi/4}$

Figure 7 shows the effect of a rotation $r_{\pi/2}$.

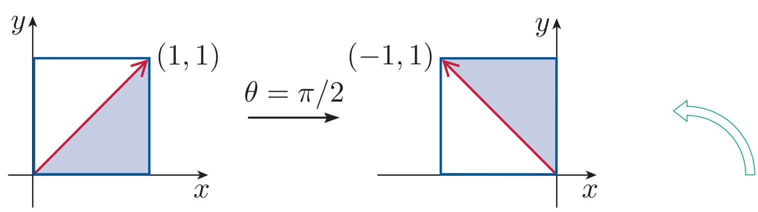


Figure 7 A rotation $r_{\pi/2}$

A **reflection** q_ϕ of $\mathbb{R}^2$ reflects vectors in the straight line through the origin that makes an angle ϕ with the x-axis (measured anticlockwise).

Figure 8 shows the effect of reflection $q_{\pi/4}$.

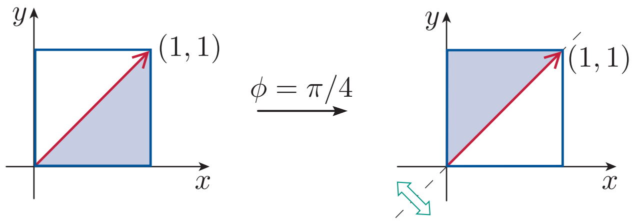


Figure 8 A reflection $q_{\pi/4}$

Figure 9 shows the effect of a reflection $q_{\pi/2}$.

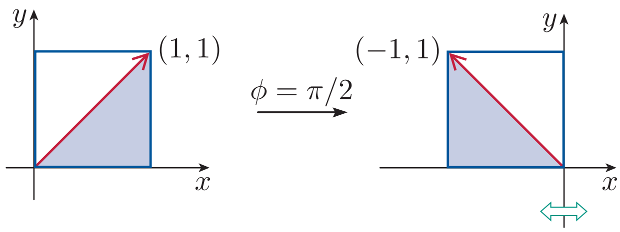


Figure 9 A reflection $q_{\pi/2}$

Exercise C82

For each of the following functions, draw a diagram to show the effect of the function on the rectangle with corners at $(0,0)$, $(2,0)$, $(2,1)$ and $(0,1)$, and on the vector $(2,1)$. State whether the function is a dilation, a scaling, a rotation or a reflection.

(a) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (2x, 3y)$

(b) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (x, -y)$

(c) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (-y, x)$

We now use matrix multiplication, from Unit C1, to obtain algebraic definitions of the four types of function defined geometrically above: dilation, scaling, rotation and reflection.

A k-dilation of $\mathbb{R}^2$ maps (x, y) to (kx, ky). This can be represented by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix}.$$

A (k, l)-scaling of $\mathbb{R}^2$ maps (x, y) to (kx, ly). This can be represented by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} k & 0 \\ 0 & l \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ ly \end{pmatrix}.$$

An algebraic definition for a rotation r_θ of $\mathbb{R}^2$ can be obtained by considering Figure 10, where r_θ maps (x, y) to (x', y').

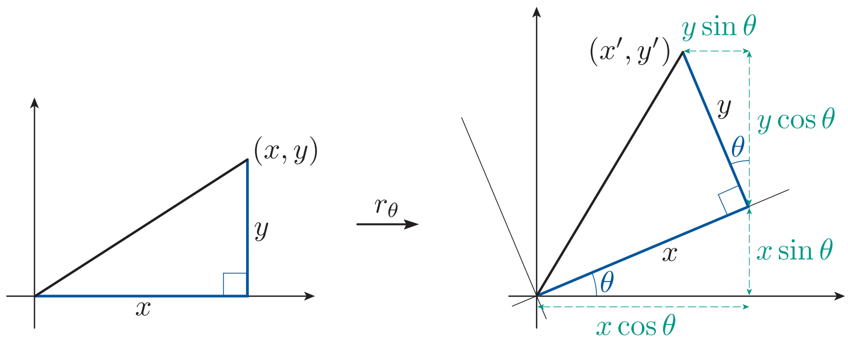


Figure 10 A rotation r_θ (through an angle of θ).

It can be seen that

$$(x, y) \longmapsto (x', y') = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

This can be represented by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}.$$

For example, $r_{\pi/6}$ can be represented by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2}x - \frac{1}{2}y \\ \frac{1}{2}x + \frac{\sqrt{3}}{2}y \end{pmatrix}.$$

Similarly, it can be shown (you will show this in Exercise C88) that a reflection q_ϕ of $\mathbb{R}^2$ can be defined algebraically by

$$(x, y) \longmapsto (x\cos 2\phi + y\sin 2\phi, x\sin 2\phi - y\cos 2\phi).$$

This can be represented by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos 2\phi + y\sin 2\phi \\ x\sin 2\phi - y\cos 2\phi \end{pmatrix}.$$

For example, $q_{\pi/6}$ can be represented by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + \frac{\sqrt{3}}{2}y \\ \frac{\sqrt{3}}{2}x - \frac{1}{2}y \end{pmatrix}.$$

We have seen that each of the four types of function can be represented by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

for some real numbers a, b, c and d.

The existence of a matrix representation is not the only property shared by these functions of the plane: they also share several striking geometric properties. In each of the examples, the image of the unit square is either a square or a rectangle; each of these functions maps straight lines to straight lines – indeed, each maps parallel lines to parallel lines. Any function that maps parallel lines to parallel lines will map parallelograms to parallelograms. Another geometric property shared by these four functions is that they also all map the origin to itself.

Figure 11 shows the effect of a general transformation t on two vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$, where t maps parallelograms to parallelograms and preserves the origin.

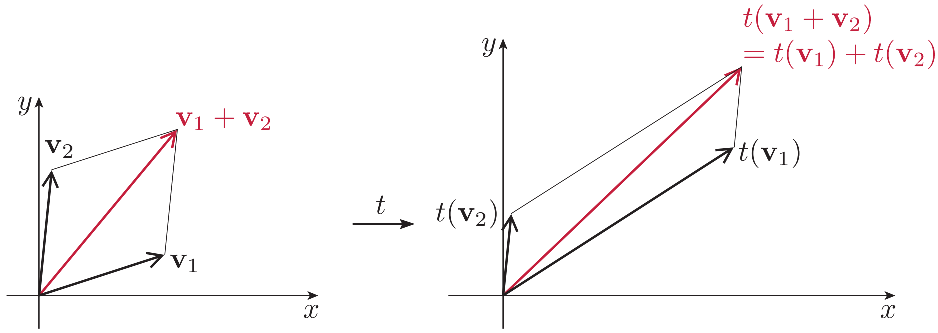


Figure 11 Parallelograms are mapped to parallelograms

Bearing in mind the Parallelogram Law for addition of vectors from Unit A1 *Sets, functions and vectors*, this illustrates that for each function t in one of the four classes above, we have

$$t(\mathbf{v}_1 + \mathbf{v}_2) = t(\mathbf{v}_1) + t(\mathbf{v}_2), \quad \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2.$$

Such a function t also preserves scalar multiples, as illustrated in Figure 12; that is, if $\alpha\mathbf{v}$ is a scalar multiple of a vector $\mathbf{v}$, then the image of $\alpha\mathbf{v}$ under t is a scalar multiple of the image of $\mathbf{v}$ under t.

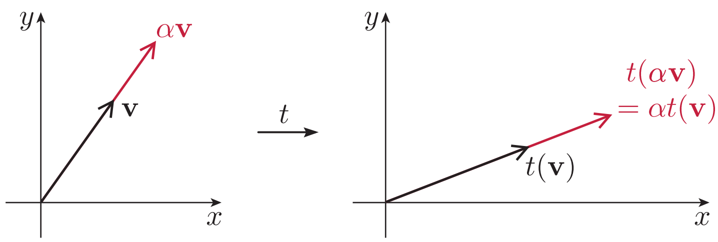


Figure 12 Scalar multiples are preserved

We have

$$t(\alpha\mathbf{v}) = \alpha\, t(\mathbf{v}), \quad \text{for all } \mathbf{v} \in \mathbb{R}^2,\ \alpha \in \mathbb{R}.$$

We use these two algebraic properties to define a *linear transformation* from any vector space to another: a linear transformation is any function from a vector space V to a vector space W that has these two algebraic properties. You will see why these functions are called *linear* transformations in Subsection 1.3.

Definition

Let V and W be vector spaces. A function $t : V \longrightarrow W$ is a **linear transformation** if it satisfies the following properties.

LT1 $t(\mathbf{v}_1 + \mathbf{v}_2) = t(\mathbf{v}_1) + t(\mathbf{v}_2)$, for all $\mathbf{v}_1, \mathbf{v}_2 \in V$.

LT2 $t(\alpha\mathbf{v}) = \alpha\, t(\mathbf{v})$, for all $\mathbf{v} \in V$, $\alpha \in \mathbb{R}$.

In Section 2 we show that the functions between finite-dimensional vector spaces that have these two properties are precisely those functions that have matrix representations.

Suppose that $t : V \longrightarrow W$ is a linear transformation. It follows from property LT1 that if we know the images of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ under t, then we can find the image of the vector $\mathbf{v}_1 + \mathbf{v}_2$. It follows from property LT2 that if we know the image of a vector $\mathbf{v}$ under t, then we can find the image of any scalar multiple of $\mathbf{v}$.

Thus, once we know the images of some vectors, we can find the images of more vectors by applying properties LT1 and LT2. In fact, if we know the image of each vector in a *basis* for V, then we can find the image of *every* vector in V. It is this property that makes linear transformations so important; we will prove it at the end of this section.

All the functions of the plane that we have studied map the origin to itself. In fact, any linear transformation $t : V \longrightarrow W$ maps the zero vector of V to the zero vector of W. To see this, we use property LT2:

$$t(\mathbf{0}) = t(0\mathbf{0}) = 0t(\mathbf{0}) = \mathbf{0}.$$

We have proved the following result.

Theorem C37

Let $t : V \longrightarrow W$ be a linear transformation. Then $t(\mathbf{0}) = \mathbf{0}$.

It follows from Theorem C37 that a function $t : V \longrightarrow W$ where $t(\mathbf{0}) \neq \mathbf{0}$ is *not* a linear transformation; for example, the function

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (y - 1, x) \end{aligned}$$

is not a linear transformation because

$$t(\mathbf{0}) = t(0, 0) = (-1, 0) \neq \mathbf{0}.$$

However, a function t with the property $t(\mathbf{0}) = \mathbf{0}$ is not necessarily a linear transformation. For example, the function

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x, |y|) \end{aligned}$$

satisfies $t(\mathbf{0}) = \mathbf{0}$ but is not a linear transformation. To see this, consider the two vectors $(0, 1)$ and $(0, -1)$. LT1 is not satisfied because

$$t\left((0, 1) + (0, -1)\right) = t(0, 0) = (0, 0)$$

and

$$t(0, 1) + t(0, -1) = (0, 1) + (0, 1) = (0, 2).$$

LT2 is also not satisfied; this can be shown, for example, by taking the vector $(0, 1)$ and $\alpha = -1$.

The following strategy can be used to test whether a given function is a linear transformation.

Strategy C14

To determine whether or not a given function $t : V \longrightarrow W$ is a linear transformation, do the following.

1. Check whether $t(\mathbf{0}) = \mathbf{0}$; if not, then t is not a linear transformation.
2. Check whether t satisfies the following two properties.

   LT1 $\quad t(\mathbf{v}_1 + \mathbf{v}_2) = t(\mathbf{v}_1) + t(\mathbf{v}_2), \quad$ for all $\mathbf{v}_1, \mathbf{v}_2 \in V$.

   LT2 $\quad t(\alpha\mathbf{v}) = \alpha\, t(\mathbf{v}), \quad$ for all $\mathbf{v} \in V$, $\alpha \in \mathbb{R}$.

The function t is a linear transformation if and only if both these properties are satisfied.

You may have noticed that if the two properties in step 2 of the strategy both hold, then t is a linear transformation and we do not also need to check step 1. We have, however, included step 1 in the strategy as this can provide a quick way of showing that some functions are *not* linear transformations. On the other hand, if step 1 holds but either one of properties LT1 or LT2 fails, then you do not need to check the other.

Worked Exercise C49

Use Strategy C14 to determine whether or not each of the following functions is a linear transformation.

(a) $\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (2x, y) \end{aligned}$

(b) $\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto ((x + y)^2, y^2) \end{aligned}$

Solution

(a) You may notice that t is a $(2, 1)$-scaling, and so expect it to be a linear transformation.

Here $t(\mathbf{0}) = \mathbf{0}$, so t may be a linear transformation.

Next we check whether t satisfies LT1:

$$t(\mathbf{v}_1 + \mathbf{v}_2) = t(\mathbf{v}_1) + t(\mathbf{v}_2), \quad \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2.$$

In $\mathbb{R}^2$, let $\mathbf{v}_1 = (x_1, y_1)$ and $\mathbf{v}_2 = (x_2, y_2)$. Then

$$\begin{aligned} t(\mathbf{v}_1 + \mathbf{v}_2) &= t(x_1 + x_2, y_1 + y_2) \\ &= (2(x_1 + x_2), y_1 + y_2) \\ &= (2x_1 + 2x_2, y_1 + y_2) \end{aligned}$$

and

$$\begin{aligned} t(\mathbf{v}_1) + t(\mathbf{v}_2) &= (2x_1, y_1) + (2x_2, y_2) \\ &= (2x_1 + 2x_2, y_1 + y_2). \end{aligned}$$

These expressions are equal, so LT1 is satisfied.

Finally, we check whether t satisfies LT2:

$$t(\alpha \mathbf{v}) = \alpha\, t(\mathbf{v}), \quad \text{for all } \mathbf{v} \in \mathbb{R}^2,\ \alpha \in \mathbb{R}.$$

Let $\mathbf{v} = (x, y)$ be a vector in $\mathbb{R}^2$ and let $\alpha \in \mathbb{R}$. Then

$$t(\alpha \mathbf{v}) = t(\alpha x, \alpha y) = (2\alpha x, \alpha y)$$

and

$$\alpha\, t(\mathbf{v}) = \alpha\, t(x, y) = \alpha(2x, y) = (2\alpha x, \alpha y).$$

These expressions are equal, so LT2 is satisfied.

Since LT1 and LT2 are satisfied, t is a linear transformation.

(b) Here $t(\mathbf{0}) = \mathbf{0}$, so t may be a linear transformation.

Next we check whether t satisfies LT1:

$$t(\mathbf{v}_1 + \mathbf{v}_2) = t(\mathbf{v}_1) + t(\mathbf{v}_2), \quad \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2.$$

In $\mathbb{R}^2$, let $\mathbf{v}_1 = (x_1, y_1)$ and $\mathbf{v}_2 = (x_2, y_2)$. Then

$$\begin{aligned} t(\mathbf{v}_1 + \mathbf{v}_2) &= t(x_1 + x_2, y_1 + y_2) \\ &= ((x_1 + x_2 + y_1 + y_2)^2, (y_1 + y_2)^2) \end{aligned}$$

and

$$\begin{aligned} t(\mathbf{v}_1) + t(\mathbf{v}_2) &= ((x_1 + y_1)^2, y_1^2) + ((x_2 + y_2)^2, y_2^2) \\ &= ((x_1 + y_1)^2 + (x_2 + y_2)^2, y_1^2 + y_2^2). \end{aligned}$$

These expressions are not equal in general, so LT1 is not satisfied.

Thus t is not a linear transformation.

Since property LT1 is not satisfied, there is no need to check property LT2; however, in this case it also does not hold.

Exercise C83

Use Strategy C14 to determine whether or not each of the following functions is a linear transformation.

(a) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (x + 3y, y)$

(b) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (x + 2, y + 1)$

In Exercise C83(a) you showed that the function

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x + 3y, y) \end{aligned}$$

is a linear transformation. This function is an example of a *shear*, or *skew*, of $\mathbb{R}^2$.

As illustrated in Figure 13, in general, a **shear** of $\mathbb{R}^2$ in the x-direction by a factor k is the linear transformation

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x + ky, y). \end{aligned}$$

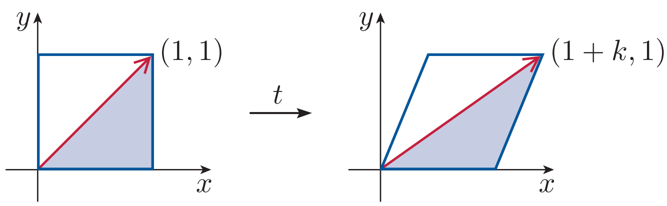


Figure 13 A shear in the x-direction by a factor of k

In Exercise C83(b) you showed that the function

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x + 2, y + 1) \end{aligned}$$

is not a linear transformation. This function is an example of a *translation* of $\mathbb{R}^2$.

As illustrated in Figure 14, in general, a **translation** of $\mathbb{R}^2$ by (a, b) is the function

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x + a, y + b). \end{aligned}$$

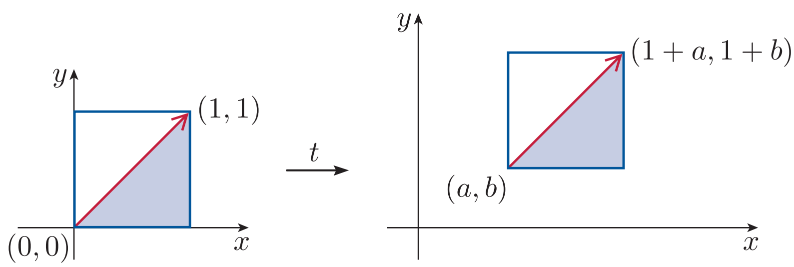


Figure 14 A translation by (a, b)

A translation is not a linear transformation unless $a = b = 0$, since otherwise it does not map the origin to itself.

1.2 Examples of linear transformations

You have seen many examples of functions from $\mathbb{R}^2$ to $\mathbb{R}^2$. In general, given any two vector spaces V and W, we can define functions from V to W. For example, consider the function t from $\mathbb{R}^3$ to $\mathbb{R}^2$ that projects each vector in $\mathbb{R}^3$ onto the (x, y)-plane, as illustrated in Figure 15:

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ (x, y, z) &\longmapsto (x, y). \end{aligned}$$

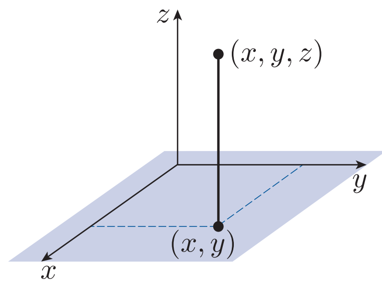


Figure 15 A projection from $\mathbb{R}^3$ onto the (x, y)-plane

This function is a linear transformation as shown in the next worked exercise.

Worked Exercise C50

Show that the following function t from $\mathbb{R}^3$ to $\mathbb{R}^2$ is a linear transformation.

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ (x, y, z) &\longmapsto (x, y) \end{aligned}$$

Solution

Note that the question says 'show', not 'determine'; we know that it *is* a linear transformation. Thus we use the definition rather than Strategy C14 and avoid the need to check whether $t(\mathbf{0}) = \mathbf{0}$.

First we show that t satisfies LT1:

$$t(\mathbf{v}_1 + \mathbf{v}_2) = t(\mathbf{v}_1) + t(\mathbf{v}_2), \quad \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3.$$

In $\mathbb{R}^3$, let $\mathbf{v}_1 = (x_1, y_1, z_1)$ and $\mathbf{v}_2 = (x_2, y_2, z_2)$. Then

$$\begin{aligned} t(\mathbf{v}_1 + \mathbf{v}_2) &= t(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= (x_1 + x_2, y_1 + y_2) \end{aligned}$$

and

$$\begin{aligned} t(\mathbf{v}_1) + t(\mathbf{v}_2) &= t(x_1, y_1, z_1) + t(x_2, y_2, z_2) \\ &= (x_1, y_1) + (x_2, y_2) \\ &= (x_1 + x_2, y_1 + y_2). \end{aligned}$$

These expressions are equal, so LT1 is satisfied.

Next we show that t satisfies LT2:

$$t(\alpha\mathbf{v}) = \alpha\, t(\mathbf{v}), \quad \text{for all } \mathbf{v} \in \mathbb{R}^3,\ \alpha \in \mathbb{R}.$$

Let $\mathbf{v} = (x, y, z)$ be a vector in $\mathbb{R}^3$ and let $\alpha \in \mathbb{R}$. Then

$$t(\alpha\mathbf{v}) = t(\alpha x, \alpha y, \alpha z) = (\alpha x, \alpha y)$$

and

$$\alpha\, t(\mathbf{v}) = \alpha\, t(x, y, z) = \alpha(x, y) = (\alpha x, \alpha y).$$

These expressions are equal, so LT2 is satisfied.

Since LT1 and LT2 are satisfied, t is a linear transformation.

Worked Exercise C51

Determine whether or not the following function is a linear transformation.

$$\begin{aligned} t : \mathbb{R}^4 &\longrightarrow \mathbb{R}^2 \\ (x, y, z, w) &\longmapsto (xy, z) \end{aligned}$$

Solution

The question says 'determine', so here we do use the strategy.

We use Strategy C14.

Since $t(\mathbf{0}) = \mathbf{0}$, t may be a linear transformation.

Next we check whether t satisfies LT1:

$$t(\mathbf{v}_1 + \mathbf{v}_2) = t(\mathbf{v}_1) + t(\mathbf{v}_2), \quad \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^4.$$

In $\mathbb{R}^4$, let $\mathbf{v}_1 = (x_1, y_1, z_1, w_1)$ and $\mathbf{v}_2 = (x_2, y_2, z_2, w_2)$. Then

$$\begin{aligned} t(\mathbf{v}_1 + \mathbf{v}_2) &= t(x_1 + x_2, y_1 + y_2, z_1 + z_2, w_1 + w_2) \\ &= ((x_1 + x_2)(y_1 + y_2), z_1 + z_2) \end{aligned}$$

and

$$\begin{aligned} t(\mathbf{v}_1) + t(\mathbf{v}_2) &= (x_1 y_1, z_1) + (x_2 y_2, z_2) \\ &= (x_1 y_1 + x_2 y_2, z_1 + z_2). \end{aligned}$$

Since $(x_1 + x_2)(y_1 + y_2) \neq x_1 y_1 + x_2 y_2$ in general, LT1 is not satisfied.

Thus t is not a linear transformation.

Exercise C84

Determine whether or not each of the following functions is a linear transformation.

(a) $\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^4 \\ (x, y) &\longmapsto (x, y, x, y) \end{aligned}$

(b) $\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ (x, y, z) &\longmapsto x^2 \end{aligned}$

(c) $\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^4 \\ (x, y, z) &\longmapsto (x, y, z, 1) \end{aligned}$

In the previous subsection we gave an algebraic definition of a rotation of $\mathbb{R}^2$. Similarly, a rotation of $\mathbb{R}^3$ in an anticlockwise direction about the z-axis through an angle θ, as illustrated in Figure 16, is given by

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z). \end{aligned}$$

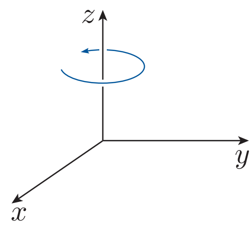


Figure 16 A rotation about the z-axis through an angle θ

Exercise C85

Show that the following function t is a linear transformation.

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z). \end{aligned}$$

So far we have considered functions $t : V \longrightarrow W$ where $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ for some $m, n \in \mathbb{N}$. There are, however, many functions between other types of vector space.

Recall from Unit C2 that the vector space P_n is the set of all polynomials of degree less than n, so

$$\begin{aligned} P_3 &= \{p(x) : p(x) = a + bx + cx^2,\ a, b, c \in \mathbb{R}\}, \\ P_2 &= \{p(x) : p(x) = a + bx,\ a, b \in \mathbb{R}\}. \end{aligned}$$

Worked Exercise C52

Consider the function that maps each polynomial $p(x) = a + bx + cx^2$ in P_3 to its derivative $p'(x) = b + 2cx$ in P_2:

$$\begin{aligned} t : P_3 &\longrightarrow P_2 \\ p(x) &\longmapsto p'(x). \end{aligned}$$

Determine whether or not this function is a linear transformation.

Solution

We use Strategy C14.

Since the zero element of P_3 is $p(x) = 0$, we have $p'(x) = 0$ and thus $t(\mathbf{0}) = \mathbf{0}$; so t may be a linear transformation.

Next we check whether t satisfies LT1:

$$t(p(x) + q(x)) = t(p(x)) + t(q(x)), \quad \text{for all } p(x), q(x) \in P_3.$$

Let $p(x), q(x) \in P_3$. Then

$$t(p(x) + q(x)) = (p(x) + q(x))' = p'(x) + q'(x)$$

and

$$t(p(x)) + t(q(x)) = p'(x) + q'(x).$$

These expressions are equal, so LT1 is satisfied.

Finally, we check whether t satisfies LT2:

$$t(\alpha p(x)) = \alpha\, t(p(x)), \quad \text{for all } p(x) \in P_3,\ \alpha \in \mathbb{R}.$$

Let $p(x) \in P_3$ and $\alpha \in \mathbb{R}$. Then

$$t(\alpha p(x)) = (\alpha p(x))' = \alpha p'(x)$$

and

$$\alpha\, t(p(x)) = \alpha p'(x).$$

These expressions are equal, so LT2 is satisfied.

Since LT1 and LT2 are satisfied, t is a linear transformation.

Exercise C86

Consider the function t from P_3 to itself obtained by adding to each polynomial $p(x) = a + bx + cx^2$ in P_3 the number $p(2) = a + 2b + 4c$:

$$\begin{aligned} t : P_3 &\longrightarrow P_3 \\ p(x) &\longmapsto p(x) + p(2). \end{aligned}$$

Determine whether or not this function is a linear transformation.

There are also linear transformations of infinite-dimensional vector spaces. For example, let V be the vector space of all real functions. An argument similar to that in the solution to Exercise C86 shows that the following function is a linear transformation:

$$\begin{aligned} t : V &\longrightarrow V \\ f(x) &\longmapsto f(x) + f(2). \end{aligned}$$

Zero transformation

Since every vector space contains a zero vector, given any two vector spaces V and W, there is a particularly simple function mapping each vector in V to the zero vector in W:

$$\begin{aligned} t : V &\longrightarrow W \\ \mathbf{v} &\longmapsto \mathbf{0}. \end{aligned}$$

This function is a linear transformation. To show this, we first show that t satisfies LT1:

$$t(\mathbf{v}_1 + \mathbf{v}_2) = t(\mathbf{v}_1) + t(\mathbf{v}_2), \quad \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in V.$$

Let $\mathbf{v}_1, \mathbf{v}_2 \in V$. Then $\mathbf{v}_1 + \mathbf{v}_2$ is also in V, so

$$t(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{0}$$

and

$$t(\mathbf{v}_1) + t(\mathbf{v}_2) = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

So LT1 is satisfied.

Next we show that t satisfies LT2:

$$t(\alpha\mathbf{v}) = \alpha\, t(\mathbf{v}), \quad \text{for all } \mathbf{v} \in V,\ \alpha \in \mathbb{R}.$$

Let $\mathbf{v} \in V$ and $\alpha \in \mathbb{R}$. Then $\alpha\mathbf{v}$ is also in V, so

$$t(\alpha\mathbf{v}) = \mathbf{0}$$

and

$$\alpha\, t(\mathbf{v}) = \alpha\mathbf{0} = \mathbf{0}.$$

So LT2 is satisfied.

Since LT1 and LT2 are satisfied, t is a linear transformation.

Definition

The **zero transformation** from V to W is the linear transformation

$$\begin{aligned} t : V &\longrightarrow W \\ \mathbf{v} &\longmapsto \mathbf{0}. \end{aligned}$$

Identity transformation

Given a vector space V, there is another particularly simple function, this time from V to itself, mapping each vector in V to itself:

$$\begin{aligned} i_V : V &\longrightarrow V \\ \mathbf{v} &\longmapsto \mathbf{v}. \end{aligned}$$

Exercise C87

Show that the function i_V is a linear transformation.

Definition

The **identity transformation** of V is the linear transformation

$$\begin{aligned} i_V : V &\longrightarrow V \\ \mathbf{v} &\longmapsto \mathbf{v}. \end{aligned}$$

We omit the subscript V when the vector space is clear from the context.

1.3 Linear combinations of vectors

Recall from Unit C2 that a linear combination of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is an expression of the form $\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n$, where $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. We end this section by proving that linear combinations of vectors are preserved under a linear transformation; that is, if $\mathbf{v}$ is a given linear combination of vectors $\mathbf{v}_i$, then the image of $\mathbf{v}$ is the same linear combination of the images of the vectors $\mathbf{v}_i$. This explains why these functions are called linear transformations. In fact, some texts use this theorem as the definition of a linear transformation.

Theorem C38

A function $t: V \longrightarrow W$ is a linear transformation if and only if it satisfies

LT3 $t(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2) = \alpha_1 t(\mathbf{v}_1) + \alpha_2 t(\mathbf{v}_2)$,

for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ and all $\alpha_1, \alpha_2 \in \mathbb{R}$.

Proof We start by proving the 'only if' part using LT1 and LT2 to show that a linear transformation satisfies LT3.

If a function $t: V \longrightarrow W$ is a linear transformation, then it satisfies LT1 and LT2. We show that this implies that it satisfies LT3.

Let $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. Then it follows from LT1 that

$$t(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2) = t(\alpha_1\mathbf{v}_1) + t(\alpha_2\mathbf{v}_2),$$

and from LT2 that

$$t(\alpha_1\mathbf{v}_1) + t(\alpha_2\mathbf{v}_2) = \alpha_1 t(\mathbf{v}_1) + \alpha_2 t(\mathbf{v}_2).$$

So t satisfies the property LT3:

$$t(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2) = \alpha_1 t(\mathbf{v}_1) + \alpha_2 t(\mathbf{v}_2), \quad \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in V \text{ and all } \alpha_1, \alpha_2 \in \mathbb{R}.$$

We now prove the 'if' part using property LT3 to show that LT1 and LT2 are satisfied.

Suppose that a function $t: V \longrightarrow W$ satisfies property LT3. Then it also satisfies LT1 and LT2, since

$$t(\mathbf{v}_1 + \mathbf{v}_2) = t(\mathbf{v}_1) + t(\mathbf{v}_2), \quad \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in V,$$

is a special case of LT3 with $\alpha_1 = \alpha_2 = 1$, and

$$t(\alpha\mathbf{v}) = \alpha\, t(\mathbf{v}), \quad \text{for all } \mathbf{v} \in V,\ \alpha \in \mathbb{R},$$

is a special case of LT3 with $\mathbf{v}_2 = \mathbf{0}$, $\mathbf{v}_1 = \mathbf{v}$, $\alpha_1 = \alpha$ and $\alpha_2 = 0$.

Thus a function is a linear transformation if and only if it satisfies property LT3. ■

We now prove that linear combinations of any number of vectors are preserved under a linear transformation.

Theorem C39

Let $t : V \longrightarrow W$ be a linear transformation. Then

$$t(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n) = \alpha_1 t(\mathbf{v}_1) + \alpha_2 t(\mathbf{v}_2) + \cdots + \alpha_n t(\mathbf{v}_n),$$
$$\text{for all } \mathbf{v}_1, \ldots, \mathbf{v}_n \in V \text{ and all } \alpha_1, \ldots, \alpha_n \in \mathbb{R},\ n \in \mathbb{N}.$$

Proof We use proof by mathematical induction as in Unit A3 *Mathematical language and proof* and start by writing out clearly what we take $P(n)$ to be.

Let $P(n)$ be the statement

$$t(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n) = \alpha_1 t(\mathbf{v}_1) + \alpha_2 t(\mathbf{v}_2) + \cdots + \alpha_n t(\mathbf{v}_n),$$
$$\text{for all } \mathbf{v}_1, \ldots, \mathbf{v}_n \in V \text{ and all } \alpha_1, \ldots, \alpha_n \in \mathbb{R}.$$

Next, we carry out step 1; that is, we check that $P(1)$ holds.

Since t is a linear transformation, LT2 is satisfied, so

$$t(\alpha_1\mathbf{v}_1) = \alpha_1 t(\mathbf{v}_1), \quad \text{for all } \mathbf{v}_1 \in V,\ \alpha_1 \in \mathbb{R}.$$

Thus $P(1)$ is true.

Now we proceed with step 2. We start by stating clearly our assumption, $P(k)$.

We assume that $P(k)$ is true for some positive integer k; that is,

$$t(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k) = \alpha_1 t(\mathbf{v}_1) + \alpha_2 t(\mathbf{v}_2) + \cdots + \alpha_k t(\mathbf{v}_k),$$
$$\text{for all } \mathbf{v}_1, \ldots, \mathbf{v}_k \in V \text{ and all } \alpha_1, \ldots, \alpha_k \in \mathbb{R}.$$

We state clearly our desired conclusion, $P(k+1)$.

We wish to deduce that $P(k+1)$ is true; that is,

$$\begin{aligned} &t(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k + \alpha_{k+1}\mathbf{v}_{k+1}) \\ &\quad = \alpha_1 t(\mathbf{v}_1) + \alpha_2 t(\mathbf{v}_2) + \cdots + \alpha_k t(\mathbf{v}_k) + \alpha_{k+1} t(\mathbf{v}_{k+1}). \end{aligned}$$

Now, $\mathbf{v}_1, \ldots, \mathbf{v}_{k+1} \in V$ and all $\alpha_1, \ldots, \alpha_{k+1} \in \mathbb{R}$. We have

$$\begin{aligned} &t(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k + \alpha_{k+1}\mathbf{v}_{k+1}) \\ &\quad = t((\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k) + \alpha_{k+1}\mathbf{v}_{k+1}) \\ &\quad = t(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k) + t(\alpha_{k+1}\mathbf{v}_{k+1}) \quad \text{(by LT1)} \\ &\quad = t(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k) + \alpha_{k+1} t(\mathbf{v}_{k+1}) \quad \text{(by LT2)} \\ &\quad = \alpha_1 t(\mathbf{v}_1) + \alpha_2 t(\mathbf{v}_2) + \cdots + \alpha_k t(\mathbf{v}_k) + \alpha_{k+1} t(\mathbf{v}_{k+1}) \quad \text{(by } P(k)\text{)}. \end{aligned}$$

We have proved that $P(k) \Rightarrow P(k+1)$.

Thus $P(k) \Rightarrow P(k+1)$, for $k = 1, 2, \ldots$.

Hence, by the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$. ■

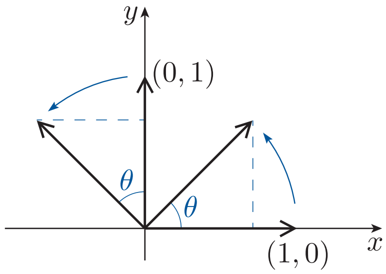


Figure 17 The rotation r_θ

Theorem C39 is an important result. It means that, given a linear transformation $t : V \longrightarrow W$ and the images of each of the vectors in a basis for V, we can determine the image of *any* vector in V.

Consider the linear transformation r_θ that rotates each vector in $\mathbb{R}^2$ anticlockwise through an angle θ about the origin, as illustrated in Figure 17. The standard basis for $\mathbb{R}^2$ is $\{(1,0),(0,1)\}$. From Figure 17, we can check that

$$r_\theta(1,0) = (\cos\theta, \sin\theta) \quad \text{and} \quad r_\theta(0,1) = (-\sin\theta, \cos\theta).$$

We now write each vector (x,y) in $\mathbb{R}^2$ in the form

$$(x,y) = x(1,0) + y(0,1),$$

so, from Theorem C39,

$$\begin{aligned} r_\theta(x,y) &= r_\theta(x(1,0) + y(0,1)) \\ &= x r_\theta(1,0) + y r_\theta(0,1) \\ &= x(\cos\theta, \sin\theta) + y(-\sin\theta, \cos\theta) \\ &= (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta). \end{aligned}$$

This method of finding an algebraic definition for r_θ is simpler than the geometric approach used in Subsection 1.1 and is more generally applicable.

Exercise C88

Find the image of a vector (x,y) in $\mathbb{R}^2$ under the reflection q_ϕ, given that $q_\phi(1,0) = (\cos 2\phi, \sin 2\phi)$ and $q_\phi(0,1) = (\sin 2\phi, -\cos 2\phi)$.

2 Matrices of linear transformations

In this section you will see how the images of basis vectors can be used to find the matrix representation of a linear transformation.

2.1 Finding matrix representations

In Section 1 you met several examples of matrix representations of linear transformations. For example, you saw that a k-dilation of $\mathbb{R}^2$ can be represented by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix}$$

and a rotation r_θ of $\mathbb{R}^2$ can be represented by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}.$$

In this section we show that any linear transformation $t : V \longrightarrow W$ between finite-dimensional vector spaces has a matrix representation

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \longmapsto \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix},$$

or

$$\mathbf{v}^T \longmapsto \mathbf{A}\mathbf{v}^T = \mathbf{w}^T.$$

Matrix representations are important because they are an aid to performing calculations with linear transformations; in particular, they are easily handled by computers.

You have seen that it is sometimes convenient to use a non-standard basis $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ for a vector space V. Recall from Unit C2 that if $\mathbf{v}$ is a vector in V and

$$\mathbf{v} = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n,$$

then the numbers $v_1, \dots, v_n$ are the *coordinates* of $\mathbf{v}$ with respect to the basis E (the *E-coordinates* of $\mathbf{v}$). The *E-coordinate representation* of $\mathbf{v}$ is $\mathbf{v}_E = (v_1, \dots, v_n)_E$.

For example, let E be the basis $\{(1,1),(1,0)\}$ for $\mathbb{R}^2$. The vector $\mathbf{v} = (5,2)$ in $\mathbb{R}^2$ can be written as

$$\mathbf{v} = 2(1,1) + 3(1,0),$$

so the E-coordinate representation of $\mathbf{v}$ is

$$\mathbf{v}_E = (2,3)_E.$$

For another example, consider the basis $E = \{1 + x^2, x^2, 2 - x\}$ for the vector space P_3. As

$$1 + x + 2x^2 = 3(1 + x^2) - x^2 - (2 - x),$$

the E-coordinate representation of the polynomial $1 + x + 2x^2$ is $(3, -1, -1)_E$.

The following exercises should remind you how to write a vector in terms of its coordinates with respect to a given basis.

Exercise C89

Find the E-coordinate representation of the vector $\mathbf{v} = (3,1)$ in $\mathbb{R}^2$ for each of the following bases E for $\mathbb{R}^2$.

(a) $E = \{(3,1),(2,1)\}$ (b) $E = \{(1,2),(2,1)\}$

Exercise C90

Find the E-coordinate representation of the polynomial $p(x) = 2 + 3x$ in P_2 for each of the following bases E for P_2.

(a) $E = \{1, x\}$ (the standard basis) (b) $E = \{1, 4 + 6x\}$

(c) $E = \{2x, 1 + 4x\}$

We now define a *matrix representation* of a linear transformation between finite-dimensional vector spaces, with respect to specified bases.

Definition

Let V and W be vector spaces of dimensions n and m, respectively. Let $t : V \longrightarrow W$ be a linear transformation, let $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for V, let $F = \{\mathbf{f}_1, \ldots, \mathbf{f}_m\}$ be a basis for W and let $\mathbf{A}$ be an $m \times n$ matrix such that

$$t(\mathbf{v})_F = \mathbf{A}\mathbf{v}_E, \quad \text{for each vector } \mathbf{v} \text{ in } V.$$

Then $\mathbf{v}_E \longmapsto \mathbf{A}\mathbf{v}_E = t(\mathbf{v})_F$ is the **matrix representation** of t with respect to the bases E and F, and $\mathbf{A}$ is the **matrix** of t with respect to the bases E and F.

Remarks

1. A matrix of a linear transformation from an n-dimensional vector space to an m-dimensional vector space is an $m \times n$ matrix, not an $n \times m$ matrix as you might expect.
2. Strictly speaking, since we defined vectors as *row vectors*, we should write $\mathbf{v}_E^T \longmapsto \mathbf{A}\mathbf{v}_E^T = t(\mathbf{v})_F^T$. However, we omit the transpose symbols for simplicity, and we often write these vectors as row vectors to save space.
3. When $E = F$, we refer to the matrix representation with respect to the basis E.

Later in this section we will *prove* that there is exactly one matrix of t with respect to the bases E and F, but first we develop a strategy (Strategy C15) for finding the matrix of a linear transformation.

Matrix representations using standard bases

We start by considering linear transformations where both E and F are the standard basis.

Exercise C91

Each of the following linear transformations $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by a matrix representation with respect to the standard basis $\{(1,0),(0,1)\}$ for $\mathbb{R}^2$. In each case, find the images of the vectors $(1,0)$ and $(0,1)$. What do you notice about the relationship between the vectors $t(1,0)$ and $t(0,1)$ and the 2×2 matrix of the linear transformation?

(a) A $(3,2)$-scaling of $\mathbb{R}^2$.

$$t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(b) A rotation $r_{\pi/4}$ of $\mathbb{R}^2$.

$$t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

In Exercise C91 you saw two examples in which, given a transformation defined by a matrix, the coordinates of the images of the standard basis vectors of the domain were the columns of the matrix. It turns out that this is always the case, even for non-standard bases: the coordinates of the images of the basis vectors of the domain are the columns of the matrix. This gives a strategy for finding the matrix of a linear transformation between any two finite-dimensional vector spaces with respect to *any* bases for the domain and codomain.

Strategy C15

To find the matrix $\mathbf{A}$ of a linear transformation $t : V \longrightarrow W$ with respect to the basis $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for V, and the basis $F = \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ for W, do the following.

1. Find $t(\mathbf{e}_1), t(\mathbf{e}_2), \ldots, t(\mathbf{e}_n)$.
2. Find the F-coordinates of each of these image vectors.

$$\begin{aligned} t(\mathbf{e}_1) &= (\mathbf{a}_{11}, \mathbf{a}_{21}, \ldots, \mathbf{a}_{m1})_F \\ t(\mathbf{e}_2) &= (\mathbf{a}_{12}, \mathbf{a}_{22}, \ldots, \mathbf{a}_{m2})_F \\ &\vdots \\ t(\mathbf{e}_n) &= (\mathbf{a}_{1n}, \mathbf{a}_{2n}, \ldots, \mathbf{a}_{mn})_F \end{aligned}$$

3. Construct the matrix $\mathbf{A}$ column by column.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

We first illustrate the strategy with some exercises and then prove that it works later in this section.

Worked Exercise C53

For each of the following linear transformations t, find the matrix representation of t with respect to the standard bases for the domain and codomain.

(a) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$
$(x, y) \longmapsto (2x, 3x + y, y)$

(b) $t : P_3 \longrightarrow P_2$
$p(x) \longmapsto p'(x)$

Solution

(a) We use Strategy C15.

The standard basis for $\mathbb{R}^2$ is $\{(1, 0), (0, 1)\}$.

We find the images of the vectors in the domain basis $E = \{(1, 0), (0, 1)\}$:

$$t(1, 0) = (2, 3, 0), \quad t(0, 1) = (0, 1, 1).$$

The standard basis for $\mathbb{R}^3$ is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. There is really nothing to do here when the basis in the codomain is the standard basis since the images are already expressed with respect to this basis; we show the working here for completeness.

We find the F-coordinates of each of these image vectors, where $F = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$:

$$t(1, 0) = (2, 3, 0)_F, \quad t(0, 1) = (0, 1, 1)_F.$$

We now construct the matrix of t by writing down the coordinates of the image vectors column by column – keeping the columns in the same order as the corresponding domain basis vectors.

Hence the matrix of t with respect to the standard bases for the domain and codomain is

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 3 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus the matrix representation of t with respect to these bases is

$$\begin{pmatrix} x \\ y \end{pmatrix}_E \longmapsto \begin{pmatrix} 2 & 0 \\ 3 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}_E = \begin{pmatrix} 2x \\ 3x + y \\ y \end{pmatrix}_F.$$

We have included the subscripts E and F here, but often will omit these where the bases are the standard ones.

(b) The standard basis for P_3 is $\{1, x, x^2\}$. Therefore the three basis vectors and their derivatives are as follows: $p_1(x) = 1$ and $p_1'(x) = 0$, $p_2(x) = x$ and $p_2'(x) = 1$, $p_3(x) = x^2$ and $p_3'(x) = 2x$.

We find the images of the vectors in the domain basis $E = \{1, x, x^2\}$:

$$t(1) = 0, \quad t(x) = 1, \quad t(x^2) = 2x.$$

The standard basis for P_2 is $\{1, x\}$. We notice that $0 = 0 + 0x$, $1 = 1 + 0x$ and $2x = 0 + 2x$.

We find the F-coordinates of each of these image vectors, where $F = \{1, x\}$:

$$t(1) = (0, 0)_F, \quad t(x) = (1, 0)_F, \quad t(x^2) = (0, 2)_F.$$

We keep the columns in this order.

Hence the matrix of t with respect to the standard bases for the domain and codomain is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Thus the matrix representation of t with respect to these bases for P_3 and P_2 is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_E \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}_E = \begin{pmatrix} b \\ 2c \end{pmatrix}_F.$$

We have $t(a + bx + cx^2) = b + 2cx$.

Exercise C92

For each of the following linear transformations t, find the matrix representation of t with respect to the standard bases for the domain and codomain.

(a) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (x + 3y, y)$

(b) $t : P_3 \longrightarrow P_3$
$p(x) \longmapsto p(x) + p(2)$

(c) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$
$(x, y) \longmapsto (x, y, x, y)$

(d) $t : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$
$(x, y, z) \longmapsto (x, y)$

Matrix representations using non-standard bases

So far we have used the strategy to find matrix representations with respect to standard bases. We now use the strategy to find matrix representations with respect to other bases.

We start with a non-standard basis for the domain and the standard basis for the codomain.

Worked Exercise C54

Find the matrix representation of the linear transformation

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ (x, y, z) &\longmapsto (x, y) \end{aligned}$$

with respect to the non-standard domain basis
$E = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ and the standard codomain basis
$F = \{(1, 0), (0, 1)\}$.

Solution

We use Strategy C15.

We find the images of the vectors in the domain basis
$E = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$:

$$t(1, 1, 1) = (1, 1), \quad t(1, 1, 0) = (1, 1), \quad t(1, 0, 0) = (1, 0).$$

We find the F-coordinates of each of these image vectors, where
$F = \{(1, 0), (0, 1)\}$:

$$t(1, 1, 1) = (1, 1)_F, \quad t(1, 1, 0) = (1, 1)_F, \quad t(1, 0, 0) = (1, 0)_F.$$

We keep the columns in this order.

Hence the matrix of t with respect to the bases E and F is

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Thus the matrix representation of t with respect to the non-standard basis E for $\mathbb{R}^3$ and the standard basis F for $\mathbb{R}^2$ is

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_E \longmapsto \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_E = \begin{pmatrix} v_1 + v_2 + v_3 \\ v_1 + v_2 \end{pmatrix}_F$$

Using v_1, v_2 and v_3 instead of x, y and z helps emphasise that these are coordinates with respect to a non-standard basis.

Compare the matrix representation in Worked Example C54 to that found in Exercise C92(d) for this linear transformation with respect to the

standard basis in both the domain and the codomain:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

In general, different bases give different matrix representations.

We now consider a non-standard basis in the codomain while keeping to the standard basis in the domain.

Worked Exercise C55

Find the matrix representation of the linear transformation

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^4 \\ (x, y) &\longmapsto (x, y, x, y) \end{aligned}$$

with respect to the standard domain basis $E = \{(1,0),(0,1)\}$ and the non-standard codomain basis $F = \{(1,0,0,0),(1,1,0,0),(1,1,1,0),(1,1,1,1)\}$.

Solution

We use Strategy C15.

We find the images of the vectors in the domain basis $E = \{(1,0),(0,1)\}$:

$$t(1,0) = (1,0,1,0), \quad t(0,1) = (0,1,0,1).$$

We now write these image vectors in terms of their coordinates with respect to the codomain basis – this requires some work!

We find the F-coordinates of each of these image vectors, where $F = \{(1,0,0,0),(1,1,0,0),(1,1,1,0),(1,1,1,1)\}$.

For the first image vector, we need $a, b, c, d \in \mathbb{R}$ such that

$$(1,0,1,0) = (a,b,c,d)_F.$$

Since

$$\begin{aligned} (a,b,c,d)_F &= a(1,0,0,0) + b(1,1,0,0) + c(1,1,1,0) + d(1,1,1,1) \\ &= (a+b+c+d, b+c+d, c+d, d), \end{aligned}$$

by equating coordinates we obtain the following system

$$\begin{aligned} a + b + c + d &= 1 \\ b + c + d &= 0 \\ c + d &= 1 \\ d &= 0. \end{aligned}$$

Solving, we have $d = 0$, $c = 1$, $b = -1$ and $a = 1$, so

$$(1,0,1,0) = (1,-1,1,0)_F.$$

Therefore

$$t(1,0) = (1,-1,1,0)_F.$$

We have found the F-coordinates of the image of the first domain basis vector. If the equations had been more difficult to solve we could have used Gauss–Jordan elimination as we did in Unit C1.

For the second image vector, we need $e, f, g, h \in \mathbb{R}$ such that

$$(0,1,0,1) = (e,f,g,h)_F.$$

Since

$$\begin{aligned}(e,f,g,h)_F &= e(1,0,0,0) + f(1,1,0,0) + g(1,1,1,0) + h(1,1,1,1)\\ &= (e+f+g+h, f+g+h, g+h, h),\end{aligned}$$

by equating coordinates we obtain the system

$$\begin{aligned} e + f + g + h &= 0\\ f + g + h &= 1\\ g + h &= 0\\ h &= 1.\end{aligned}$$

Solving, we have $h = 1$, $g = -1$, $f = 1$ and $e = -1$, so

$$(0,1,0,1) = (-1,1,-1,1)_F.$$

Therefore

$$t(0,1) = (-1,1,-1,1)_F.$$

We keep the columns in this order.

Hence the matrix of t with respect to the bases E and F is

$$\mathbf{A} = \begin{pmatrix} 1 & -1\\ -1 & 1\\ 1 & -1\\ 0 & 1\end{pmatrix}.$$

Thus the matrix representation of t with respect to the standard basis E for $\mathbb{R}^2$ and the non-standard basis F for $\mathbb{R}^4$ is

$$\begin{pmatrix} v_1\\ v_2\end{pmatrix}_E \longmapsto \begin{pmatrix} 1 & -1\\ -1 & 1\\ 1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix} v_1\\ v_2\end{pmatrix}_E = \begin{pmatrix} v_1 - v_2\\ -v_1 + v_2\\ v_1 - v_2\\ v_2\end{pmatrix}_F.$$

Compare the matrix representation in Worked Exercise C55 to that found in Exercise C92(c) for this linear transformation with respect to the

standard bases in both the domain and codomain:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ x \\ y \end{pmatrix}.$$

Finally, we look at an example with non-standard bases for both the domain and the codomain.

Worked Exercise C56

Find the matrix representation of the linear transformation

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ (x, y) &\longmapsto (2x, 3x + y, y) \end{aligned}$$

with respect to the non-standard domain basis $E = \{(1, 1), (1, 0)\}$ and the non-standard codomain basis $F = \{(1, 1, 1), (0, 1, 1), (0, 0, 1))\}$.

Solution

We use Strategy C15.

We find the images of the domain basis vectors $E = \{(1, 1), (1, 0)\}$:

$$t(1, 1) = (2, 4, 1), \quad t(1, 0) = (2, 3, 0).$$

We find the F-coordinates of each of these image vectors, where $F = \{(1, 1, 1), (0, 1, 1), (0, 0, 1))\}$.

For the first image vector we need $a, b, c \in \mathbb{R}$ such that

$$(2, 4, 1) = (a, b, c)_F.$$

Since

$$\begin{aligned} (a, b, c)_F &= a(1, 1, 1) + b(0, 1, 1) + c(0, 0, 1) \\ &= (a, a + b, a + b + c), \end{aligned}$$

by equating coordinates we obtain the system

$$\begin{aligned} a \qquad\;\; &= 2 \\ a + b \quad\; &= 4 \\ a + b + c &= 1. \end{aligned}$$

Solving, we have $a = 2$, $b = 2$ and $c = -3$, so

$$(2, 4, 1) = (2, 2, -3)_F.$$

Therefore

$$t(1, 1) = (2, 2, -3)_F.$$

For the second image vector we need $d, e, f \in \mathbb{R}$ such that

$$(2, 3, 0) = (d, e, f)_F.$$

Since

$$\begin{aligned}(d,e,f)_F &= d(1,1,1)+e(0,1,1)+f(0,0,1)\\ &= (d,d+e,d+e+f),\end{aligned}$$

by equating coordinates we obtain the following system

$$\begin{aligned} d \qquad\quad &= 2\\ d+e \qquad &= 3\\ d+e+f &= 0.\end{aligned}$$

Solving, we have $d=2$, $e=1$ and $f=-3$, so

$$(2,3,0)=(2,1,-3)_F.$$

Therefore

$$t(1,0)=(2,1,-3)_F.$$

Hence the matrix of t with respect to the bases E and F is

$$\mathbf{A}=\begin{pmatrix}2 & 2\\ 2 & 1\\ -3 & -3\end{pmatrix}.$$

Thus the matrix representation of t with respect to the non-standard basis E for $\mathbb{R}^2$ and the non-standard basis F for $\mathbb{R}^3$ is

$$\begin{pmatrix}v_1\\ v_2\end{pmatrix}_E \longmapsto \begin{pmatrix}2 & 2\\ 2 & 1\\ -3 & -3\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}_E = \begin{pmatrix}2v_1+2v_2\\ 2v_1+v_2\\ -3v_1-3v_2\end{pmatrix}_F.$$

Compare the matrix representation in Worked Exercise C56 to that found in Worked Exercise C53(a) for this linear transformation with respect to the standard bases in both the domain and codomain:

$$\begin{pmatrix}x\\ y\end{pmatrix} \longmapsto \begin{pmatrix}2 & 0\\ 3 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}2x\\ 3x+y\\ y\end{pmatrix}.$$

The following exercise involves both standard and non-standard bases in the domain and codomain.

Exercise C93

Find the matrix of the linear transformation

$$\begin{aligned} t:\mathbb{R}^3 &\longrightarrow \mathbb{R}^2\\ (x,y,z) &\longmapsto (x,y)\end{aligned}$$

with respect to each of the following bases E for $\mathbb{R}^3$ and F for $\mathbb{R}^2$.

(a) $E=\{(1,0,1),(1,0,0),(1,1,1)\}$
$F=\{(1,0),(0,1)\}$ (standard basis for $\mathbb{R}^2$)

(b) $E = \{(1,0,0),(0,1,0),(0,0,1)\}$ (standard basis for $\mathbb{R}^3$)
$F = \{(2,1),(1,1)\}$

(c) $E = \{(0,1,0),(1,1,1),(0,1,1)\}$ $\quad F = \{(1,3),(2,4)\}$

You have seen that a linear transformation $t : V \longrightarrow W$ has different matrix representations depending on the bases used for the domain and codomain. Moreover, the order of the elements in a basis is important. For example, in the next exercise you should obtain different matrices for t for each part: although the bases contain the same elements, the order in which they appear in the domain basis is different.

In summary, note the following two facts.

- *Different bases* for V and W give *different* matrix representations.
- A *different order* of basis elements gives a *different* matrix representation.

Exercise C94

Find the matrix representation of the linear transformation

$$\begin{aligned} t : P_3 &\longrightarrow P_2 \\ p(x) &\longmapsto p'(x) \end{aligned}$$

with respect to each of the following bases E for P_3 and F for P_2.

(a) $E = \{1, x, x^2\}$ $\quad F = \{2x, 1+x\}$

(b) $E = \{x, x^2, 1\}$ $\quad F = \{2x, 1+x\}$

The unique matrix representation of a linear transformation

You have seen that the matrix representation of a linear transformation depends on the bases for both the domain and the codomain, and the order of these basis elements. However, for given ordered basis elements, there is precisely *one* matrix representation: the one given by Strategy C15. Using the notation in the strategy, the unique matrix representation of a linear transformation t with respect to the bases E and F is

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}_E \longmapsto \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}_E = \begin{pmatrix} a_{11}v_1 + \cdots + a_{1n}v_n \\ a_{21}v_1 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + \cdots + a_{mn}v_n \end{pmatrix}_F .$$

We now prove this result. If you are short of time, you should skim through this proof and come back to it when time permits.

Theorem C40

Let $t : V \longrightarrow W$ be a linear transformation, let $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for V and let $F = \{\mathbf{f}_1, \ldots, \mathbf{f}_m\}$ be a basis for W. Let

$$\begin{aligned} t(\mathbf{e}_1) &= (a_{11}, a_{21}, \ldots, a_{m1})_F, \\ t(\mathbf{e}_2) &= (a_{12}, a_{22}, \ldots, a_{m2})_F, \\ &\vdots \\ t(\mathbf{e}_n) &= (a_{1n}, a_{2n}, \ldots, a_{mn})_F. \end{aligned}$$

Then there is exactly one matrix of t with respect to the bases E and F, namely

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Proof We start by showing that $\mathbf{A}$ is a matrix of t with respect to the (ordered) bases E and F.

Suppose that the conditions of the theorem are satisfied and that $(v_1, \ldots, v_n)_E$ is the E-coordinate representation of a vector $\mathbf{v} \in V$. Then we have

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n.$$

By Theorem C39, linear transformations preserve linear combinations of vectors, so

$$\begin{aligned} t(\mathbf{v}) &= t(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n) \\ &= v_1 t(\mathbf{e}_1) + v_2 t(\mathbf{e}_2) + \cdots + v_n t(\mathbf{e}_n) \\ &= v_1(a_{11}, a_{21}, \ldots, a_{m1})_F + v_2(a_{12}, a_{22}, \ldots, a_{m2})_F + \cdots \\ &\quad + v_n(a_{1n}, a_{2n}, \ldots, a_{mn})_F \\ &= (v_1a_{11} + \cdots + v_na_{1n},\ v_1a_{21} + \cdots + v_na_{2n},\ \ldots, \\ &\qquad v_1a_{m1} + \cdots + v_na_{mn})_F. \end{aligned}$$

So the first coordinate of $t(\mathbf{v})$ is $a_{11}v_1 + \cdots + a_{1n}v_n$, the second coordinate of $t(\mathbf{v})$ is $a_{21}v_1 + \cdots + a_{2n}v_n$, and so on. These coordinates can be obtained by matrix multiplication as follows

$$\begin{pmatrix} a_{11}v_1 + \cdots + a_{1n}v_n \\ a_{21}v_1 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + \cdots + a_{mn}v_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}_E \longmapsto \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}_E = \begin{pmatrix} a_{11}v_1 + \cdots + a_{1n}v_n \\ a_{21}v_1 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + \cdots + a_{mn}v_n \end{pmatrix}_F$$

is a matrix representation of t with respect to the bases E and F, and

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is a matrix of t with respect to the bases E and F.

We now show that $\mathbf{A}$ is the *only* possible matrix of t with respect to the (ordered) bases E and F. We do this by assuming that there is another possible matrix $\mathbf{B}$ and concluding that $\mathbf{B}$ must be equal to $\mathbf{A}$.

Suppose that $\mathbf{B}$ is also a matrix of t with respect to the bases E and F where

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}.$$

Since $\mathbf{e}_1$ is the first basis vector in E, we have $\mathbf{e}_1 = (1, 0, \ldots, 0)_E$, and the image of $\mathbf{e}_1$ under t is

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_E \longmapsto \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_E = \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{pmatrix}_E;$$

that is,

$$t(\mathbf{e}_1) = (b_{11}, b_{21}, \ldots, b_{m1})_F.$$

However,

$$t(\mathbf{e}_1) = (a_{11}, a_{21}, \ldots, a_{m1})_F,$$

so the first column of $\mathbf{B}$ is equal to the first column of $\mathbf{A}$.

Similarly, we find that

$$\begin{aligned} t(\mathbf{e}_2) &= (b_{12}, b_{22}, \ldots, b_{m2})_F = (a_{12}, a_{22}, \ldots, a_{m2})_F, \\ t(\mathbf{e}_3) &= (b_{13}, b_{23}, \ldots, b_{m3})_F = (a_{13}, a_{23}, \ldots, a_{m3})_F, \\ &\vdots \qquad\qquad\qquad\qquad \vdots \\ t(\mathbf{e}_n) &= (b_{1n}, b_{2n}, \ldots, b_{mn})_F = (a_{1n}, a_{2n}, \ldots, a_{mn})_F. \end{aligned}$$

Therefore each subsequent column of $\mathbf{B}$ is also the same as the corresponding column of $\mathbf{A}$. Since $\mathbf{A}$ and $\mathbf{B}$ are both $m \times n$ matrices and their corresponding entries are equal, we have $\mathbf{B} = \mathbf{A}$.

Thus $\mathbf{A}$ is the only matrix of t with respect to the bases E and F. ■

2.2 An equivalent definition

We have shown that any linear transformation $t : V \longrightarrow W$, where V and W are finite-dimensional vector spaces, has a matrix representation. We now show the converse – that a function that has a matrix representation is a linear transformation. We will use the following result about matrix multiplication: if $\mathbf{A}$ and $\mathbf{B}$ are matrices and α a scalar, then $(\alpha\mathbf{A})\mathbf{B} = \mathbf{A}(\alpha\mathbf{B})$, whenever this product exists. You might like to prove this result yourself; it is included as a 'challenging' exercise in the additional exercises booklet for this unit.

Theorem C41

Let $t : V \longrightarrow W$ be a function that has a matrix representation. Then t is a linear transformation.

Proof Suppose that the function $t : V \longrightarrow W$ has a matrix representation

$$\mathbf{v}_E \longmapsto \mathbf{A}\mathbf{v}_E = t(\mathbf{v})_F.$$

We first show that t satisfies LT1: that for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ we have $t(\mathbf{v}_1 + \mathbf{v}_2) = t(\mathbf{v}_1) + t(\mathbf{v}_2)$.

Let $\mathbf{v}_1, \mathbf{v}_2 \in V$. Then

$$t(\mathbf{v}_1 + \mathbf{v}_2)_F = \mathbf{A}(\mathbf{v}_1 + \mathbf{v}_2)_E$$

and

$$\begin{aligned} t(\mathbf{v}_1)_F + t(\mathbf{v}_2)_F &= \mathbf{A}(\mathbf{v}_1)_E + \mathbf{A}(\mathbf{v}_2)_E \\ &= \mathbf{A}(\mathbf{v}_1 + \mathbf{v}_2)_E, \end{aligned}$$

by the distributive property for matrix multiplication.

So $t(\mathbf{v}_1 + \mathbf{v}_2)_F = t(\mathbf{v}_1)_F + t(\mathbf{v}_2)_F$, and hence $t(\mathbf{v}_1 + \mathbf{v}_2) = t(\mathbf{v}_1) + t(\mathbf{v}_2)$, because the F-coordinate representation of a vector is unique. Therefore the function t satisfies LT1.

We now show that t satisfies LT2: that for all $\mathbf{v} \in V$ and $\alpha \in \mathbb{R}$ we have $t(\alpha\mathbf{v}) = \alpha\, t(\mathbf{v})$.

Let $\mathbf{v} \in V$ and $\alpha \in \mathbb{R}$. Then

$$\alpha\, t(\mathbf{v})_F = \alpha\mathbf{A}\mathbf{v}_E$$

and

$$t(\alpha\mathbf{v})_F = \mathbf{A}(\alpha\mathbf{v})_E = \alpha\mathbf{A}\mathbf{v}_E,$$

by the result about matrix multiplication quoted above.

So $t(\alpha\mathbf{v})_F = \alpha\, t(\mathbf{v})_F$, and hence $t(\alpha\mathbf{v}) = \alpha\, t(\mathbf{v})$, because the F-coordinate representation of a vector is unique. Therefore the function t also satisfies LT2.

Since both LT1 and LT2 are satisfied, the function t is a linear transformation. ■

Theorems C40 and C41 imply the following.

Corollary C42

A function $t : V \longrightarrow W$, where V and W are finite-dimensional vector spaces, is a linear transformation if and only if it has a matrix representation.

This means, for example, that the linear transformations from $\mathbb{R}^2$ to itself are those functions that have a matrix representation

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

So the linear transformations from $\mathbb{R}^2$ to itself are those functions of the form

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (ax + by, cx + dy) \end{aligned} \tag{1}$$

for some $a, b, c, d \in \mathbb{R}$.

Similar expressions exist for linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Exercise C95

Use the linear transformation form (1) to determine which of the following functions are linear transformations.

(a) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (y, 2x + y)$

(b) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (x^2, y)$

(c) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (x, 2xy + y)$

(d) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (3x, x + 4y)$

3 Composition and invertibility

In this section you will use the matrix representation of a linear transformation to find composite linear transformations and investigate properties of linear transformations, such as invertibility.

3.1 Composition Rule

In the previous section you saw that a function $t : V \longrightarrow W$, where V and W are finite-dimensional vector spaces, is a linear transformation if and only if it has a matrix representation. We now use some of the properties of matrices that you met in Unit C1 to develop our understanding of linear transformations.

We begin by considering the composition of linear transformations. The composite of two functions $t : V \longrightarrow W$ and $s : W \longrightarrow X$ is

$$\begin{aligned} s \circ t : V &\longrightarrow X \\ \mathbf{v} &\longmapsto s(t(\mathbf{v})), \end{aligned}$$

as shown in Figure 18.

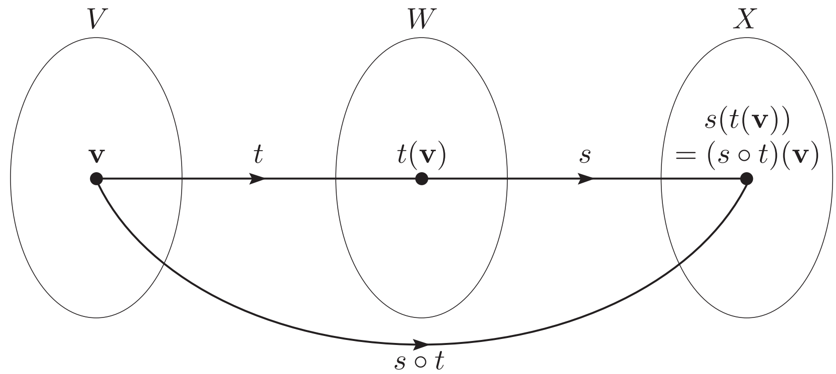


Figure 18 The composite $s \circ t$

Consider the linear transformations

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x + 2y, y) \end{aligned} \quad \text{and} \quad \begin{aligned} s : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (5x, x + y). \end{aligned} \tag{2}$$

Let $(x, y) \in \mathbb{R}^2$. Then

$$t(x, y) = (x + 2y, y),$$

so

$$\begin{aligned} s(t(x, y)) &= s(x + 2y, y) \\ &= (5(x + 2y), (x + 2y) + y) \\ &= (5x + 10y, x + 3y). \end{aligned}$$

Thus the composite function $s \circ t$ is the linear transformation

$$\begin{aligned} s \circ t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (5x + 10y, x + 3y). \end{aligned}$$

In general, for linear transformations s and t from a vector space to itself, the composite functions $s \circ t$ and $t \circ s$ are not the same, as you will see in the following exercise.

Exercise C96

Let p and r be the linear transformations

$$\begin{aligned} p : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (3x + y, -x) \end{aligned} \quad \text{and} \quad \begin{aligned} r : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x, x + y). \end{aligned}$$

Find the following composite functions.

(a) $r \circ p$ (b) $p \circ r$

Each of the composite functions in Exercise C96 is a linear transformation, since it has the correct form (1). In the next theorem (Theorem C43) we show that composition of two linear transformations always gives a linear transformation.

At the beginning of this subsection we showed that the two linear transformations s and t in equation (2) can be composed to give the linear transformation

$$\begin{aligned} s \circ t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (5x + 10y, x + 3y). \end{aligned}$$

Using Strategy C15, we obtain the matrix representations of these three linear transformations with respect to the standard basis for $\mathbb{R}^2$:

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\longmapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ y \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} s : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\longmapsto \begin{pmatrix} 5 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x \\ x + y \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} s \circ t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\longmapsto \begin{pmatrix} 5 & 10 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x + 10y \\ x + 3y \end{pmatrix}. \end{aligned}$$

We can check that

$$\begin{pmatrix} 5 & 10 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

so, in this example,

$$\begin{pmatrix} \text{matrix} \\ \text{of } s \circ t \end{pmatrix} = \begin{pmatrix} \text{matrix} \\ \text{of } s \end{pmatrix} \begin{pmatrix} \text{matrix} \\ \text{of } t \end{pmatrix}.$$

We now show that this relationship between the matrices of $s \circ t$, s and t holds in general; that is, that composition of linear transformations corresponds to matrix multiplication. If you are short of time, you should just look at the structure of this proof and come back to it when time permits; part (a) checks the properties LT1 and LT2, and part (b) constructs the matrix of the composite linear transformation. To help visualise what is going on, the composite $s \circ t$, vector spaces and bases are shown in Figure 19.

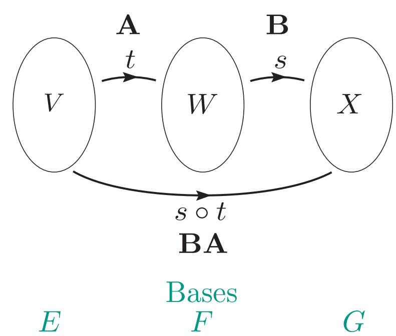


Figure 19 The composite $s \circ t$ showing the vector spaces and bases

Theorem C43 Composition Rule

Let $t : V \longrightarrow W$ and $s : W \longrightarrow X$ be linear transformations. Then:

(a) $s \circ t : V \longrightarrow X$ is a linear transformation

(b) if $\mathbf{A}$ is the matrix of t with respect to the bases E and F, and $\mathbf{B}$ is the matrix of s with respect to the bases F and G, then $\mathbf{BA}$ is the matrix of $s \circ t$ with respect to the bases E and G.

Proof Let $t : V \longrightarrow W$ and $s : W \longrightarrow X$ be linear transformations.

(a) We first show that $s \circ t$ satisfies LT1: that for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ we have $(s \circ t)(\mathbf{v}_1 + \mathbf{v}_2) = (s \circ t)(\mathbf{v}_1) + (s \circ t)(\mathbf{v}_2)$.

Let $\mathbf{v}_1, \mathbf{v}_2 \in V$. Then, since t and s both satisfy LT1, we have

$$\begin{aligned}(s \circ t)(\mathbf{v}_1 + \mathbf{v}_2) &= s(t(\mathbf{v}_1 + \mathbf{v}_2)) \\ &= s(t(\mathbf{v}_1) + t(\mathbf{v}_2)) \\ &= s(t(\mathbf{v}_1)) + s(t(\mathbf{v}_2)).\end{aligned}$$

We also have

$$(s \circ t)(\mathbf{v}_1) + (s \circ t)(\mathbf{v}_2) = s(t(\mathbf{v}_1)) + s(t(\mathbf{v}_2)).$$

So $(s \circ t)(\mathbf{v}_1 + \mathbf{v}_2) = (s \circ t)(\mathbf{v}_1) + (s \circ t)(\mathbf{v}_2)$. Therefore the composite $s \circ t$ satisfies LT1.

We now show that $s \circ t$ satisfies LT2: that for all $\mathbf{v} \in V$, $\alpha \in \mathbb{R}$ we have $(s \circ t)(\alpha\mathbf{v}) = \alpha(s \circ t)(\mathbf{v})$.

Let $\mathbf{v} \in V$ and $\alpha \in \mathbb{R}$. Then, since t and s both satisfy LT2, we have

$$(s \circ t)(\alpha\mathbf{v}) = s(t(\alpha\mathbf{v})) = s(\alpha\, t(\mathbf{v})) = \alpha\, s(t(\mathbf{v})).$$

We also have

$$\alpha(s \circ t)(\mathbf{v}) = \alpha\, s(t(\mathbf{v})).$$

So $(s \circ t)(\alpha\mathbf{v}) = \alpha(s \circ t)(\mathbf{v})$. Therefore the composite $s \circ t$ satisfies LT2.

Since both LT1 and LT2 are satisfied, the composite $s \circ t$ is a linear transformation.

(b) Suppose that the vector spaces V, W and X have dimensions n, m and p, respectively. Then $\mathbf{A}$ is an $m \times n$ matrix of the form

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

and $\mathbf{B}$ is a $p \times m$ matrix of the form

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pm} \end{pmatrix}.$$

We use Strategy C15 to find the matrix of the linear transformation $s \circ t$ with respect to the bases E and G.

We find the images under $s \circ t$ of the vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ that form the basis E for V.

To find the image of the basis vector $\mathbf{e}_1$, we use the $n \times 1$ column matrix containing the coordinates of $\mathbf{e}_1$ with respect to the basis E. This matrix has 1 in the first row and 0 elsewhere. Using the matrix representations of t and s, we find that

$$t : \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_E \longmapsto \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_E = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}_F$$

and

$$\begin{aligned} s : \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}_F &\longmapsto \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pm} \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}_F \\ &= \begin{pmatrix} b_{11}a_{11} + \cdots + b_{1m}a_{m1} \\ b_{21}a_{11} + \cdots + b_{2m}a_{m1} \\ \vdots \\ b_{p1}a_{11} + \cdots + b_{pm}a_{m1} \end{pmatrix}_G . \end{aligned}$$

So

$$(s \circ t)(\mathbf{e}_1) = (b_{11}a_{11} + \cdots + b_{1m}a_{m1}, \dots, b_{p1}a_{11} + \cdots + b_{pm}a_{m1})_G.$$

Similarly, we find that, for $k = 2, \dots, n$,

$$(s \circ t)(\mathbf{e}_k) = (b_{11}a_{1k} + \cdots + b_{1m}a_{mk}, \dots, b_{p1}a_{1k} + \cdots + b_{pm}a_{mk})_G.$$

Next, we find the G-coordinates of each of the image vectors, but the image vectors are already in this form.

We now construct the matrix of $s \circ t$, column by column. The first column contains the coordinates of $(s \circ t)(\mathbf{e}_1)$, the second column contains the coordinates of $(s \circ t)(\mathbf{e}_2)$, and so on. Thus the matrix of $s \circ t$ with respect to the bases E and G is

$$\begin{pmatrix} b_{11}a_{11}+\cdots+b_{1m}a_{m1} & \cdots & b_{11}a_{1k}+\cdots+b_{1m}a_{mk} & \cdots & b_{11}a_{1n}+\cdots+b_{1m}a_{mn} \\ \vdots & & \vdots & & \vdots \\ b_{j1}a_{11}+\cdots+b_{jm}a_{m1} & \cdots & b_{j1}a_{1k}+\cdots+b_{jm}a_{mk} & \cdots & b_{j1}a_{1n}+\cdots+b_{jm}a_{mn} \\ \vdots & & \vdots & & \vdots \\ b_{p1}a_{11}+\cdots+b_{pm}a_{m1} & \cdots & b_{p1}a_{1k}+\cdots+b_{pm}a_{mk} & \cdots & b_{p1}a_{1n}+\cdots+b_{pm}a_{mn} \end{pmatrix}.$$

Using the rules for matrix multiplication, we find that the above matrix is the same as the matrix product $\mathbf{BA}$.

Thus $\mathbf{BA}$ is the matrix of $s \circ t$ with respect to the bases E and G. ■

Worked Exercise C57

Use the Composition Rule to find the matrix representation of the linear transformation $s \circ t$ with respect to the standard bases for the domain and codomain.

$$t : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad \text{and} \qquad s : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solution

Note that the codomain of t is $\mathbb{R}^2$: when you multiply a 2×3 matrix by a 3×1 one, the result is 2×1.

It follows from the Composition Rule that the matrix of $s \circ t$ with respect to the standard bases for the domain and codomain is

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 6 \\ 8 & 7 & 9 \end{pmatrix}.$$

Thus the matrix representation of $s \circ t$ with respect to the standard bases for the domain and codomain is

$$\begin{aligned} s \circ t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} &\longmapsto \begin{pmatrix} 2 & 3 & 6 \\ 8 & 7 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} 2x + 3y + 6z \\ 8x + 7y + 9z \end{pmatrix}. \end{aligned}$$

To work out the domain and codomain of $s \circ t$, recall that if $t : V \longrightarrow W$ and $s : W \longrightarrow X$ then $s \circ t : V \longrightarrow X$ (see Figure 18).

Exercise C97

Use the Composition Rule to find the matrix representation of the linear transformation $s \circ t$ with respect to the standard bases for the domain and codomain.

$$t : \mathbb{R}^4 \longrightarrow \mathbb{R}^2 \qquad \text{and} \qquad s : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 2 & 4 \\ 2 & 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

We now return to two examples of linear transformations of vector spaces of polynomials:

$$\begin{aligned} t : P_3 &\longrightarrow P_3 \\ p(x) &\longmapsto p(x) + p(2) \end{aligned}$$

and

$$\begin{aligned} s : P_3 &\longrightarrow P_2 \\ p(x) &\longmapsto p'(x). \end{aligned}$$

We compose these linear transformations as follows:

$$\begin{aligned} (s \circ t)(p(x)) &= s(t(p(x))) \\ &= s(p(x) + p(2)) \\ &= (p(x) + p(2))' \\ &= p'(x). \end{aligned}$$

Thus the composite is

$$\begin{aligned} s \circ t : P_3 &\longrightarrow P_2 \\ p(x) &\longmapsto p'(x). \end{aligned}$$

In this case, the functions $s \circ t$ and s are the same function.

Exercise C98

Use the Composition Rule to find the matrix representation of the linear transformation $s \circ t$ with respect to the standard bases $E = \{1, x, x^2\}$ for P_3 and $F = \{1, x\}$ for P_2, when

$$\begin{aligned} s : P_3 &\longrightarrow P_2 \\ p(x) &\longmapsto p'(x) \end{aligned} \quad \text{and} \quad \begin{aligned} t : P_3 &\longrightarrow P_3 \\ p(x) &\longmapsto p(x) + p(2). \end{aligned}$$

(In Worked Exercise C53(b) and Exercise C92(b) you found that s and t have the following matrix representations

$$s : P_3 \longrightarrow P_2$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_E \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}_E = \begin{pmatrix} b \\ 2c \end{pmatrix}_F$$

and

$$t : P_3 \longrightarrow P_3$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_E \longmapsto \begin{pmatrix} 2 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}_E = \begin{pmatrix} 2a + 2b + 4c \\ b \\ c \end{pmatrix}_E$$

with respect to the standard bases E and F for P_3 and P_2, respectively.)

In Subsection 3.2 of Unit C1 we claimed that multiplication of matrices is associative. We now prove this result, by using the Composition Rule (Theorem C43).

Corollary C44

Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be matrices of sizes $q \times p$, $p \times m$ and $m \times n$, respectively. Then

$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}.$$

Proof Let t, s and r be the linear transformations whose matrix representations with respect to the standard bases for the domain and codomain are

$$\begin{aligned} t : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ \mathbf{v} &\longmapsto \mathbf{Cv}, \end{aligned} \qquad \begin{aligned} s : \mathbb{R}^m &\longrightarrow \mathbb{R}^p \\ \mathbf{v} &\longmapsto \mathbf{Bv} \end{aligned} \quad \text{and} \quad \begin{aligned} r : \mathbb{R}^p &\longrightarrow \mathbb{R}^q \\ \mathbf{v} &\longmapsto \mathbf{Av}. \end{aligned}$$

It follows from the Composition Rule that $\mathbf{A}(\mathbf{BC})$ is the matrix of the linear transformation $r \circ (s \circ t)$ and that $(\mathbf{AB})\mathbf{C}$ is the matrix of the linear transformation $(r \circ s) \circ t$, with respect to the standard bases for the domain and codomain. The linear transformations $r \circ (s \circ t)$ and $(r \circ s) \circ t$ are equal, since $(r \circ (s \circ t))(\mathbf{v})$ and $((r \circ s) \circ t)(\mathbf{v})$ both mean $r(s(t(\mathbf{v})))$. It follows that $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$. ■

This result illustrates how we can prove results about matrices by using linear transformations. We can also prove results about linear transformations by using matrices, as we do in the next subsection.

3.2 Invertible linear transformations

In this subsection we introduce the notion of an *invertible linear transformation*. Suppose that $t : V \longrightarrow W$ is a linear transformation that is one-to-one (no two elements of V have the same image) and is also onto (the image set $t(V)$ is the whole of W); that is, each element of W is the image of exactly one element of V. Then t has an inverse function t^{-1} with domain W, such that

$$t^{-1}(t(\mathbf{v})) = \mathbf{v}, \quad \text{for each } \mathbf{v} \in V,$$

and

$$t(t^{-1}(\mathbf{w})) = \mathbf{w}, \quad \text{for each } \mathbf{w} \in W;$$

that is,

$$t^{-1} \circ t = i_V \quad \text{and} \quad t \circ t^{-1} = i_W.$$

We say that t is *invertible*. This is illustrated in Figure 20.

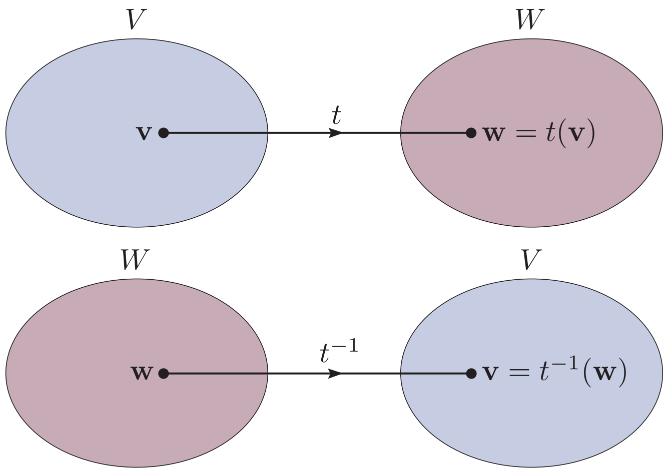


Figure 20 A linear transformation t and its inverse t^{-1}

Definition

The linear transformation $t : V \longrightarrow W$ is **invertible** if there exists an inverse function $t^{-1} : W \longrightarrow V$ such that

$$t^{-1} \circ t = i_V \quad \text{and} \quad t \circ t^{-1} = i_W.$$

Thus a linear transformation $t : V \longrightarrow W$ is invertible if and only if it is one-to-one and onto.

The linear transformation

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x, 0) \end{aligned}$$

is not invertible, since it is not one-to-one; for example,

$$t(1, 1) = t(1, 2) = (1, 0).$$

The linear transformation

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ (x, y) &\longmapsto (x, y, 0) \end{aligned}$$

is not invertible, since it is not onto: the image set $t(\mathbb{R}^2)$ is the (x, y)-plane, which is not the whole of the codomain $\mathbb{R}^3$.

Now consider the linear transformation

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (2x, 2y). \end{aligned}$$

We can check that t is one-to-one and onto and hence invertible by using the methods of Unit A1, but what is the inverse function of t?

Since t stretches each vector by a factor 2, we expect the inverse function of t to be the linear transformation

$$\begin{aligned} s : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto \left(\tfrac{1}{2}x, \tfrac{1}{2}y\right), \end{aligned}$$

which contracts each vector to half its magnitude. Since

$$s(t(x, y)) = s(2x, 2y) = (x, y)$$

and

$$t(s(x, y)) = t\left(\tfrac{1}{2}x, \tfrac{1}{2}y\right) = (x, y)$$

for each vector (x, y) in $\mathbb{R}^2$, $s \circ t$ and $t \circ s$ are both the identity transformation of $\mathbb{R}^2$, so s is the inverse function of t.

Exercise C99

Verify that the linear transformation

$$\begin{aligned} s : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x + y, 3x + 4y) \end{aligned}$$

is the inverse function of the linear transformation

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (4x - y, -3x + y). \end{aligned}$$

In fact, the inverse of any linear transformation is a linear transformation. Unfortunately, it is not always obvious whether a given linear transformation $t : V \longrightarrow W$ is invertible. Even if we know that t is one-to-one and onto and hence invertible, it may not be clear what the inverse function of t is. If V and W are both *finite*-dimensional vector spaces, however, then t has a matrix representation. The next theorem, illustrated in Figure 21, shows that this can be used to determine whether t is invertible and, if so, to find the inverse function of t. If you are short of time, you should just look at the structure of this proof and come back to it when time permits.

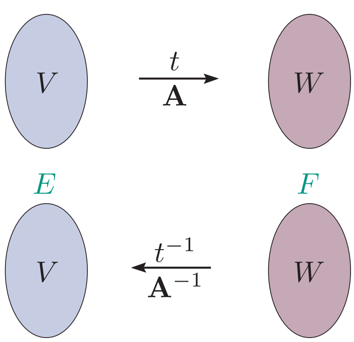


Figure 21 The linear transformation t with matrix $\mathbf{A}$, and its inverse

Theorem C45 Inverse Rule

Let $t : V \longrightarrow W$ be a linear transformation.

(a) If t is invertible, then $t^{-1} : W \longrightarrow V$ is also a linear transformation.

(b) If $\mathbf{A}$ is the matrix of t with respect to the bases E and F, then:

(i) t is invertible if and only if $\mathbf{A}$ is invertible

(ii) if t is invertible, then $\mathbf{A}^{-1}$ is the matrix of t^{-1} with respect to the bases F and E.

Proof Let $t: V \longrightarrow W$ be a linear transformation.

(a) Suppose that t is invertible.

We use Strategy C14 to show that the inverse function $t^{-1}: W \longrightarrow V$ is a linear transformation.

We first show that t^{-1} satisfies LT1: for all $\mathbf{w}_1, \mathbf{w}_2 \in W$ we have $t^{-1}(\mathbf{w}_1 + \mathbf{w}_2) = t^{-1}(\mathbf{w}_1) + t^{-1}(\mathbf{w}_2)$.

Let $\mathbf{w}_1, \mathbf{w}_2 \in W$. Then, since t is invertible and hence onto, there exist $\mathbf{v}_1, \mathbf{v}_2 \in V$ such that $\mathbf{w}_1 = t(\mathbf{v}_1)$ and $\mathbf{w}_2 = t(\mathbf{v}_2)$. Since t satisfies LT1 we have

$$\begin{aligned} t^{-1}(\mathbf{w}_1 + \mathbf{w}_2) &= t^{-1}(t(\mathbf{v}_1) + t(\mathbf{v}_2)) \\ &= t^{-1}(t(\mathbf{v}_1 + \mathbf{v}_2)) \\ &= \mathbf{v}_1 + \mathbf{v}_2. \end{aligned}$$

Also,

$$\begin{aligned} t^{-1}(\mathbf{w}_1) + t^{-1}(\mathbf{w}_2) &= t^{-1}(t(\mathbf{v}_1)) + t^{-1}(t(\mathbf{v}_2)) \\ &= \mathbf{v}_1 + \mathbf{v}_2. \end{aligned}$$

So $t^{-1}(\mathbf{w}_1 + \mathbf{w}_2) = t^{-1}(\mathbf{w}_1) + t^{-1}(\mathbf{w}_2)$. Therefore t^{-1} satisfies LT1.

Next we show that t^{-1} satisfies LT2: for all $\mathbf{w} \in W$, $\alpha \in \mathbb{R}$ we have $t^{-1}(\alpha\mathbf{w}) = \alpha\, t^{-1}(\mathbf{w})$.

Let $\mathbf{w} \in W$; then there exists $\mathbf{v} \in V$ such that $\mathbf{w} = t(\mathbf{v})$. Let $\alpha \in \mathbb{R}$; then, since t satisfies LT2 we have

$$t^{-1}(\alpha\mathbf{w}) = t^{-1}(\alpha\, t(\mathbf{v})) = t^{-1}(t(\alpha\mathbf{v})) = \alpha\mathbf{v}.$$

Also,

$$\alpha\, t^{-1}(\mathbf{w}) = \alpha\, t^{-1}(t(\mathbf{v})) = \alpha\mathbf{v}.$$

So $t^{-1}(\alpha\mathbf{w}) = \alpha\, t^{-1}(\mathbf{w})$ and LT2 is satisfied.

Since both LT1 and LT2 are satisfied, t^{-1} is a linear transformation.

(b) Let $\mathbf{A}$ be the matrix of t with respect to the bases E and F, so

$$t: \mathbf{v}_E \longmapsto \mathbf{A}\mathbf{v}_E = \mathbf{w}_F, \quad \text{for any vector } \mathbf{v} \in V.$$

We prove the 'if' statement and show that if $\mathbf{A}$ is invertible, then t is invertible. Using properties of matrices, if $\mathbf{A}$ is invertible, then $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$, where $\mathbf{I}$ is the identity matrix.

We show that if $\mathbf{A}$ is invertible, then t is invertible. Suppose that $\mathbf{A}$ is invertible. Then we know that $\mathbf{A}$ is a square matrix and $\mathbf{A}^{-1}$ is also square (and of the same size); so we can define s to be the linear transformation with the matrix representation

$$\begin{aligned} s: W &\longrightarrow V \\ \mathbf{w}_F &\longmapsto \mathbf{A}^{-1}\mathbf{w}_F = s(\mathbf{w})_E. \end{aligned}$$

We show that s is the inverse function of t, and hence that t is invertible.

It follows from the Composition Rule that $s \circ t$ has the matrix representation

$$\begin{aligned} s \circ t : V &\longrightarrow V \\ \mathbf{v}_E &\longmapsto (\mathbf{A}^{-1}\mathbf{A})\mathbf{v}_E = \mathbf{I}\mathbf{v}_E = \mathbf{v}_E. \end{aligned}$$

Thus $s(t(\mathbf{v})) = \mathbf{v}$ for each $\mathbf{v} \in V$; that is, $s \circ t = i_V$.

Similarly, it follows from the Composition Rule that $t \circ s$ has the matrix representation

$$\begin{aligned} t \circ s : W &\longrightarrow W \\ \mathbf{w}_F &\longmapsto (\mathbf{A}\mathbf{A}^{-1})\mathbf{w}_F = \mathbf{I}\mathbf{w}_F = \mathbf{w}_F. \end{aligned}$$

Thus $t(s(\mathbf{w})) = \mathbf{w}$ for each $\mathbf{w} \in W$; that is, $t \circ s = i_W$.

Since $s \circ t = i_V$ and $t \circ s = i_W$, it follows that s is the inverse function of t, so t is invertible.

We prove the 'only if' statement and show that if t is invertible, then $\mathbf{A}$ is invertible *and* $\mathbf{A}^{-1}$ is the matrix of t^{-1} with respect to the bases F and E.

We show that if t is invertible, then $\mathbf{A}$ is invertible. Suppose that t is invertible so t^{-1} is a linear transformation. Then by Theorem C40 it has a matrix representation

$$\begin{aligned} t^{-1} : W &\longrightarrow V \\ \mathbf{w}_F &\longmapsto \mathbf{B}\mathbf{w}_F = t^{-1}(\mathbf{w})_E. \end{aligned}$$

We show that $\mathbf{B} = \mathbf{A}^{-1}$.

It follows from the Composition Rule that $t^{-1} \circ t$ has the matrix representation

$$\begin{aligned} t^{-1} \circ t : V &\longrightarrow V \\ \mathbf{v}_E &\longmapsto (\mathbf{B}\mathbf{A})\mathbf{v}_E. \end{aligned}$$

Since $(t^{-1} \circ t)(\mathbf{v}) = \mathbf{v}$ for each $\mathbf{v} \in V$, it follows that

$$(\mathbf{B}\mathbf{A})\mathbf{v}_E = \mathbf{v}_E, \quad \text{for all } \mathbf{v} \in V.$$

Thus $\mathbf{B}\mathbf{A} = \mathbf{I}$.

Similarly, it follows from the Composition Rule that $t \circ t^{-1}$ has the matrix representation

$$\begin{aligned} t \circ t^{-1} : W &\longrightarrow W \\ \mathbf{w}_F &\longmapsto (\mathbf{A}\mathbf{B})\mathbf{w}_F. \end{aligned}$$

Since $(t \circ t^{-1})(\mathbf{w}) = \mathbf{w}$ for each $\mathbf{w} \in W$, it follows that

$$(\mathbf{A}\mathbf{B})\mathbf{w}_F = \mathbf{w}_F, \quad \text{for all } \mathbf{w} \in W.$$

Thus $\mathbf{A}\mathbf{B} = \mathbf{I}$.

Since

$$\mathbf{B}\mathbf{A} = \mathbf{A}\mathbf{B} = \mathbf{I},$$

it follows that $\mathbf{A}$ is invertible and $\mathbf{B} = \mathbf{A}^{-1}$. Therefore $\mathbf{A}^{-1}$ is the matrix of t^{-1} with respect to the bases F and E.

This completes the proof. ■

One consequence of the Inverse Rule is that if $t : V \longrightarrow W$ is an invertible linear transformation, then any matrix of t must be invertible and hence square. Since a matrix of t has m rows and n columns, where m is the dimension of W and n is the dimension of V, it follows that $m = n$; that is, the vector spaces V and W must have the same dimension, and we have the following corollary to Theorem C45.

Corollary C46

Let $t : V \longrightarrow W$ be an invertible linear transformation, where V and W are finite-dimensional. Then

$$\dim V = \dim W.$$

It follows that if $t : V \longrightarrow W$ is a linear transformation and V and W have *different* finite dimensions, then t is not invertible. For example, the linear transformation

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ (x, y, z) &\longmapsto (2x + y, x - y) \end{aligned}$$

is not invertible, since the domain and codomain have different dimensions.

Now suppose that $t : V \longrightarrow W$ is a linear transformation and that V and W have the *same* finite dimension. It follows from the Inverse Rule that you can use the following strategy to determine whether or not t is invertible. Recall that you saw in Subsection 5.4 of Unit C1 that a matrix is invertible if and only if its determinant is non-zero.

Strategy C16

To determine whether or not a linear transformation $t : V \longrightarrow W$ is invertible, where V and W are n-dimensional vector spaces with bases E and F, respectively, do the following.

1. Find a matrix representation of t,

$$\mathbf{v}_E \longmapsto \mathbf{A}\mathbf{v}_E = t(\mathbf{v})_F.$$

2. Evaluate $\det \mathbf{A}$.

- If $\det \mathbf{A} = 0$, then t is not invertible.
- If $\det \mathbf{A} \neq 0$, then t is invertible and $t^{-1} : W \longrightarrow V$ has the matrix representation

$$\mathbf{w}_F \longmapsto \mathbf{A}^{-1}\mathbf{w}_F = t^{-1}(\mathbf{w})_E.$$

Worked Exercise C58

Show that the following linear transformation t is invertible and find the inverse function of t.

$$t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x, y) \longmapsto (x + y, 2y)$$

Solution

We use Strategy C16 and first find a matrix representation of t.

We find a matrix representation of t using Strategy C15.

We have

$$t(1, 0) = (1, 0) \quad \text{and} \quad t(0, 1) = (1, 2).$$

Since we have the standard basis in the codomain, the F-coordinates of the image vectors are immediate.

Hence the matrix representation of t with respect to the standard bases for the domain and codomain is

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ 2y \end{pmatrix}.$$

The next step is to evaluate the determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

We have

$$\det \mathbf{A} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 1 \times 2 - 1 \times 0 = 2.$$

Since $\det \mathbf{A}$ is non-zero, t is invertible.

We now find the inverse function of t. According to Strategy C16, $t^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ has the matrix representation $\mathbf{v} \longmapsto \mathbf{A}^{-1}\mathbf{v}$, with respect to the standard bases for the domain and codomain. Since

$$\mathbf{A}^{-1} = \tfrac{1}{2} \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix},$$

it follows that t^{-1} has the matrix representation

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - \frac{1}{2}y \\ \frac{1}{2}y \end{pmatrix}.$$

So t^{-1} is the linear transformation

$$t^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x, y) \longmapsto \left(x - \tfrac{1}{2}y, \tfrac{1}{2}y\right).$$

Exercise C100

Determine which of the following linear transformations are invertible. Find the inverse function of each invertible linear transformation.

(a) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (2x + y, 4x + 2y)$

(b) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (x - y, 3x + y)$

(c) $t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$
$(x, y, z) \longmapsto (2x, 3y - x, z)$

(d) $t : P_3 \longrightarrow P_2$
$p(x) \longmapsto p'(x)$

In Worked Exercise C58 we considered the linear transformation

$$t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x, y) \longmapsto (x + y, 2y).$$

We found the matrix $\mathbf{A}$ of t with respect to the *standard* basis for $\mathbb{R}^2$, and showed that $\det \mathbf{A} = 2$. In fact, whatever bases we had chosen for the domain and codomain, we would still have obtained a matrix of t with determinant equal to 2.

It can be shown that the magnitude of the determinant of a matrix of t is the 'scaling factor' of t. Since $\det \mathbf{A} = 2$ in the above case, areas are doubled under t, as shown in Figure 22.

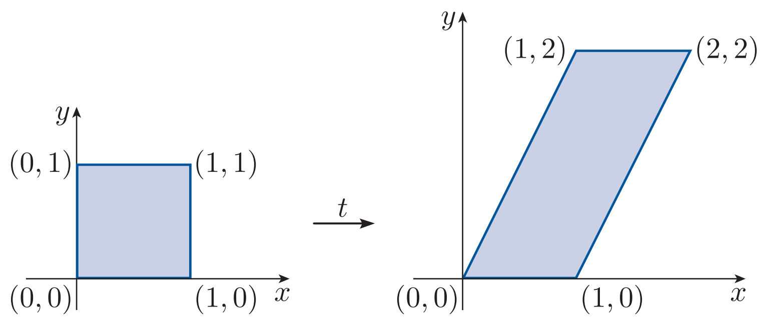


Figure 22 A linear transformation with 'scaling factor' 2

This 'scaling factor' explains the geometric interpretation of the determinant of a 2×2 matrix: that for two position vectors (a, c) and (b, d), the determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

gives the area of the parallelogram with adjacent sides given by these position vectors. The matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is the matrix of the linear transformation with respect to the standard basis for $\mathbb{R}^2$ that maps these basis vectors to (a, c) and (b, d), respectively.

For a linear transformation $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and a matrix $\mathbf{A}$ of t with $\det \mathbf{A} = 0$, the image of a unit square under t is a line or a point – these have zero area. So, in this case, t is not invertible.

3.3 Isomorphisms

You have seen that there are invertible linear transformations from $\mathbb{R}^2$ to itself, and from $\mathbb{R}^3$ to itself. In fact, whenever the vector spaces V and W have the *same* finite dimension, we can construct an invertible linear transformation from V to W.

For example, consider the two-dimensional vector spaces $\mathbb{R}^2$ and P_2. The linear transformation

$$\begin{aligned} t : P_2 &\longrightarrow \mathbb{R}^2 \\ a + bx &\longmapsto (a, b) \end{aligned}$$

is one-to-one and onto and hence invertible. By looking at a matrix representation of t in this example, we can see how to construct a general invertible linear transformation from V to W, whenever V and W have the same finite dimension.

For t above, take the standard bases $E = \{1, x\}$ for P_2 and $F = \{(1,0), (0,1)\}$ for $\mathbb{R}^2$. Then $t(1) = (1,0)$ and $t(x) = (0,1)$, so t has the matrix representation

$$\begin{pmatrix} a \\ b \end{pmatrix}_E \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}_E = \begin{pmatrix} a \\ b \end{pmatrix}_F,$$

that is,

$$\mathbf{v}_E \longmapsto \mathbf{I}_2 \mathbf{v}_E = \mathbf{w}_F.$$

More generally, let V and W be n-dimensional vector spaces, let E be a basis for V and let F be a basis for W. Then

$$\begin{aligned} t : V &\longrightarrow W \\ \mathbf{v}_E &\longmapsto \mathbf{I}_n \mathbf{v}_E = \mathbf{w}_F \end{aligned}$$

is a linear transformation from V to W. Since the identity matrix $\mathbf{I}_n$ is invertible, it follows from the Inverse Rule that t is invertible. Note that t maps the first basis vector in E to the first basis vector in F, the second basis vector in E to the second basis vector in F, and so on. We say that t is an *isomorphism* from V to W.

Definition

The vector spaces V and W are **isomorphic** if there exists an invertible linear transformation $t : V \longrightarrow W$. Such a function t is an **isomorphism**.

You met isomorphisms between groups in Unit B2 *Subgroups and isomorphisms*. Isomorphisms between vector spaces are analogous: they identify when vector spaces are 'structurally identical' to each other.

Exercise C101

Write down an isomorphism from P_3 to $\mathbb{R}^3$.

Although the examples of isomorphisms given above involve the identity matrix, any invertible linear transformation provides an isomorphism; so any invertible matrix is possible. For example, consider the following matrix $\mathbf{A}$ of a linear transformation $s : P_3 \longrightarrow \mathbb{R}^3$ with respect to the standard bases in the domain and codomain.

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

This matrix is invertible; you might like to check that it has determinant 2. It is likely that this linear transformation provides a different isomorphism between the vector spaces P_3 and $\mathbb{R}^3$ to the one you wrote down as your answer to Exercise C101.

Suppose that V and W are finite-dimensional vector spaces. We have just seen that if $\dim V = \dim W$, then there is an invertible linear transformation $t : V \longrightarrow W$; that is, V and W are isomorphic.

In particular, each n-dimensional vector space is isomorphic to $\mathbb{R}^n$.

We also know that if V and W have *different* dimensions, then there are *no* invertible linear transformations from V to W; that is, V and W are not isomorphic. Thus we have proved the following result.

Theorem C47

The finite-dimensional vector spaces V and W are isomorphic if and only if

$$\dim V = \dim W.$$

Exercise C102

State which of the following vector spaces are isomorphic to each other:

$$\mathbb{R}^2, \quad \mathbb{R}^3, \quad \mathbb{C}, \quad P_2, \quad P_3.$$

4 Image and kernel

In the previous section you saw that a linear transformation $t : V \longrightarrow W$ is invertible if t is one-to-one and onto; that is, each element of W is the image of exactly one element of V.

In this section you will meet a strategy for determining which elements of W are the images of elements of V, before investigating conditions under which an element of W is the image of more than one element of V. This enables us to prove an important result known as the *Dimension Theorem.*

Finally, we use the Dimension Theorem to show how the number of possible solutions of a system of m linear equations in n unknowns depends on the values of m and n.

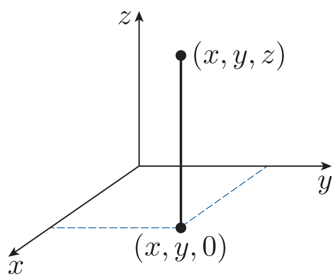


Figure 23 A projection function in $\mathbb{R}^3$

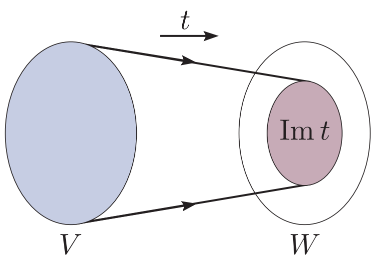


Figure 24 The image set of a linear transformation

4.1 Image of a linear transformation

Let t be the linear transformation

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (x, y, 0). \end{aligned}$$

This projects each vector (x, y, z) onto the vector $(x, y, 0)$ in the (x, y)-plane of $\mathbb{R}^3$ as shown in Figure 23.

A vector $\mathbf{w}$ in the codomain $\mathbb{R}^3$ is the image of a vector $\mathbf{v}$ in the domain $\mathbb{R}^3$ if and only if $\mathbf{w}$ is in the (x, y)-plane. We say that the (x, y)-plane is the *image set* of t. This is a two-dimensional subspace of the codomain $\mathbb{R}^3$.

Recall from Unit A1 that the image set of a function is the set of all elements of the codomain that are images of some element in the domain. Thus the image set of a linear transformation $t : V \longrightarrow W$ is the set of all vectors of W that are images of vectors of V as shown in Figure 24.

The image set of a linear transformation is sometimes simply called its *image.*

Definition

The **image set** of a linear transformation $t : V \longrightarrow W$ is the set

$$\operatorname{Im} t = \{t(\mathbf{v}) : \mathbf{v} \in V\}.$$

Note that the meaning of $\operatorname{Im} t$, which here is the image set of t, is different from that of Im used in Unit A2 *Number systems* where Im meant the imaginary part of a complex number.

It is important to remember that the image set of t is a subset of W, but it need not be equal to W because there may be some vectors of W that are not images of vectors in V. Also, some vectors of W may be images under t of more than one vector of V. Another, equivalent, way of expressing this image set is

$$\operatorname{Im} t = \{\mathbf{w} \in W : \mathbf{w} = t(\mathbf{v}),\ \text{for some } \mathbf{v} \in V\}.$$

Exercise C103

Give a geometric description of the image set of each of the following linear transformations. In each case, state whether the image set is a subspace of the codomain.

(a) $t : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$
$(x, y, z) \longmapsto (x, 0)$

(b) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (x, x)$

For each of the linear transformations in Exercise C103, the image set is a subspace of the codomain. This is true for all linear transformations.

Theorem C48

Let $t : V \longrightarrow W$ be a linear transformation. Then $\operatorname{Im} t$ is a subspace of the codomain W.

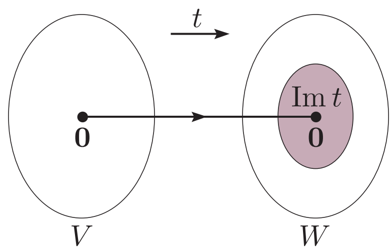


Figure 25 The image of the zero vector

Proof We follow Strategy C10 in Unit C2.

We check first that $\mathbf{0} \in \operatorname{Im} t$.

Since t is a linear transformation, $t(\mathbf{0}) = \mathbf{0}$, so $\mathbf{0} \in \operatorname{Im} t$.

This is illustrated in Figure 25.

We check next that $\operatorname{Im} t$ is closed under vector addition.

Let $\mathbf{w}_1, \mathbf{w}_2 \in \operatorname{Im} t$. Then there exist vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$ such that $\mathbf{w}_1 = t(\mathbf{v}_1)$ and $\mathbf{w}_2 = t(\mathbf{v}_2)$. Since t is a linear transformation,

$$\mathbf{w}_1 + \mathbf{w}_2 = t(\mathbf{v}_1) + t(\mathbf{v}_2) = t(\mathbf{v}_1 + \mathbf{v}_2).$$

Since V is closed under vector addition, $\mathbf{v}_1 + \mathbf{v}_2 \in V$, so $\mathbf{w}_1 + \mathbf{w}_2 \in \operatorname{Im} t$.

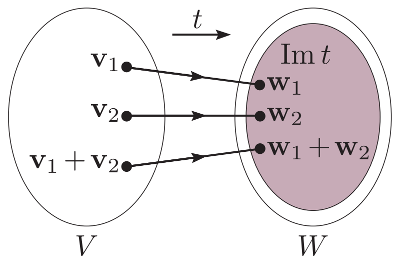


Figure 26 The image of a sum of vectors

This is illustrated in Figure 26.

Finally, we show that $\operatorname{Im} t$ is closed under scalar multiplication.

Let $\mathbf{w} \in \operatorname{Im} t$ and $\alpha \in \mathbb{R}$. Then there exists $\mathbf{v} \in V$ such that $\mathbf{w} = t(\mathbf{v})$ and, since t is a linear transformation,

$$\alpha \mathbf{w} = \alpha\, t(\mathbf{v}) = t(\alpha \mathbf{v}).$$

Since V is closed under scalar multiplication, $\alpha \mathbf{v} \in V$, so $\alpha \mathbf{w} \in \operatorname{Im} t$.

This is illustrated in Figure 27.

Thus $\operatorname{Im} t$ is a subspace of W. ∎

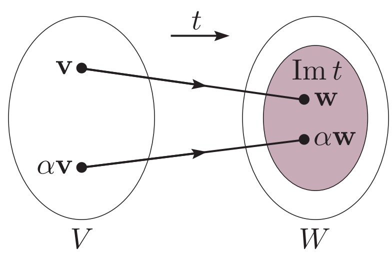


Figure 27 The image of a scalar multiple

For the linear transformations studied so far, it has been easy to write down their image sets. In general, this is not the case; so we need some way of determining the image set of a linear transformation.

If we know the image of each vector in a basis for V, then we can find the image of each vector in V since linear combinations of vectors are preserved (Theorem C39). Thus the image set of t is determined by the images of the domain basis vectors.

For example, consider the linear transformation

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (x, y, 0). \end{aligned}$$

The standard basis for the domain $\mathbb{R}^3$ is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

The images of the vectors in this basis are

$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 0).$$

These vectors all lie in $\operatorname{Im} t$, which is the (x, y)-plane, and they *span* $\operatorname{Im} t$; that is, each vector in $\operatorname{Im} t$ can be written as a linear combination of the vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 0)$.

Exercise C104

Let t be the linear transformation

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x, x). \end{aligned}$$

Determine the images of the vectors in the standard basis $\{(1, 0), (0, 1)\}$ for the domain $\mathbb{R}^2$. Do these image vectors span $\operatorname{Im} t$?

(In Exercise C103(b) you found that the image set of this linear transformation is the line $y = x$.)

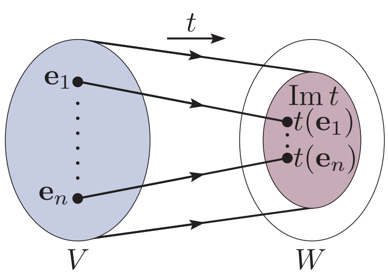


Figure 28 The images of the domain basis vectors in the image set

We now show that, for any linear transformation, the images of the domain basis vectors span the image set; the images of these domain basis vectors are illustrated in Figure 28.

Let $t : V \longrightarrow W$ be a linear transformation and let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for V. If $\mathbf{w} \in \operatorname{Im} t$, then $\mathbf{w} = t(\mathbf{v})$ for some $\mathbf{v}$ in V. Since $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis for V, there exist real numbers $v_1, \ldots, v_n$ such that

$$\mathbf{v} = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n.$$

Since t is a linear transformation, it preserves linear combinations of vectors (Theorem C39), so it follows that

$$\mathbf{w} = t(\mathbf{v}) = v_1 t(\mathbf{e}_1) + \cdots + v_n t(\mathbf{e}_n).$$

Thus $\mathbf{w}$ is a linear combination of the vectors $t(\mathbf{e}_1), \ldots, t(\mathbf{e}_n)$.

So $\{t(\mathbf{e}_1), \ldots, t(\mathbf{e}_n)\}$ is a spanning set for $\operatorname{Im} t$, as claimed.

Since a basis is a linearly independent spanning set, we now give a strategy that enables us to find a basis for the image set of a linear transformation.

Strategy C17

To find a basis for $\operatorname{Im} t$, where $t : V \longrightarrow W$ is a linear transformation, do the following.

1. Find a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for the domain V.
2. Determine the vectors $t(\mathbf{e}_1), \ldots, t(\mathbf{e}_n)$.
3. If there is a vector $\mathbf{v}$ in $S = \{t(\mathbf{e}_1), \ldots, t(\mathbf{e}_n)\}$ that is a linear combination of the other vectors in S, then discard $\mathbf{v}$ to give the set $S_1 = S - \{\mathbf{v}\}$.
4. If there is a vector $\mathbf{v}_1$ in S_1 such that $\mathbf{v}_1$ is a linear combination of the other vectors in S_1, then discard $\mathbf{v}_1$ to give the set $S_2 = S_1 - \{\mathbf{v}_1\}$.

Continue discarding vectors in this way until you obtain a linearly independent set. This set is a basis for $\operatorname{Im} t$.

Once we know a basis for the image set of a linear transformation, we know everything that we need to know about the image set; in particular, we know its dimension.

Worked Exercise C59

Let t be the linear transformation

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ (x, y, z) &\longmapsto (x + 2y + 3z, 4x + y - 2z). \end{aligned}$$

Find a basis for $\operatorname{Im} t$ and state the dimension of $\operatorname{Im} t$.

Solution

We use Strategy C17.

We take the standard basis $\{(1,0,0), (0,1,0), (0,0,1)\}$ for the domain $\mathbb{R}^3$.

We determine the images of these basis vectors:

$$t(1,0,0) = (1,4), \quad t(0,1,0) = (2,1), \quad t(0,0,1) = (3,-2).$$

The set $\{(1,4), (2,1), (3,-2)\}$ is not linearly independent since we have

$$(3,-2) = 2(2,1) - (1,4),$$

so we discard $(3,-2)$ to give the set $\{(1,4), (2,1)\}$.

We could have discarded $(1,4)$ or $(2,1)$ instead: the remaining pairs of vectors are linearly independent in each case since they are not multiples of each other.

The vectors $(1,4)$ and $(2,1)$ are linearly independent, so $\{(1,4),(2,1)\}$ is a basis for $\operatorname{Im} t$.

Since the basis has two elements, $\operatorname{Im} t$ is a two-dimensional subspace of the codomain $\mathbb{R}^2$.

In fact, since $\operatorname{Im} t$ is a two-dimensional subspace of $\mathbb{R}^2$, it is the whole of $\mathbb{R}^2$.

Exercise C105

For each of the following linear transformations t, find a basis for $\operatorname{Im} t$ and state the dimension of $\operatorname{Im} t$.

(a) $$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x, 2x+y) \end{aligned}$$

(b) $$\begin{aligned} t : P_3 &\longrightarrow P_2 \\ p(x) &\longmapsto p'(x) \end{aligned}$$

(c) $$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x,y,z) &\longmapsto (x+2y+3z, x+z, x+y+2z) \end{aligned}$$

For the linear transformation t in Exercise C105(c), $\operatorname{Im} t$ is a two-dimensional subspace of the codomain $\mathbb{R}^3$. Thus $\operatorname{Im} t$ is a plane through the origin with equation

$$ax + by + cz = 0,$$

for some $a, b, c \in \mathbb{R}$ not all zero. It is possible to use the basis that you found for $\operatorname{Im} t$ in Exercise C105(c) to work out the values of a, b and c. For example, using the basis $\{(1,1,1),(2,0,1)\}$ for $\operatorname{Im} t$, we can proceed as follows. Since the basis vectors belong to $\operatorname{Im} t$, the values a, b and c satisfy the system

$$\begin{aligned} a + b + c &= 0 \\ 2a \qquad + c &= 0. \end{aligned}$$

The second equation gives $c = -2a$. Substituting this into the first equation gives $b = a$. So $\operatorname{Im} t$ is the plane with equation

$$ax + ay - 2az = 0$$

or, equivalently,

$$x + y - 2z = 0.$$

Finally, we note that a linear transformation $t : V \longrightarrow W$ is onto when every element of W is the image of an element of V; that is, a linear transformation is onto if and only if $\operatorname{Im} t = W$. Since $\operatorname{Im} t$ is a subspace of W, if $\dim(\operatorname{Im} t) = \dim W$ then we can immediately conclude that $\operatorname{Im} t = W$ and, conversely, if $\operatorname{Im} t = W$ then $\dim(\operatorname{Im} t) = \dim W$. Thus we have the following result.

Proposition C49

A linear transformation t is onto if and only if $\dim(\operatorname{Im} t) = \dim W$.

Exercise C106

Which of the following linear transformations are onto?

(a) $$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x, 2x + y) \end{aligned}$$

(b) $$\begin{aligned} t : P_3 &\longrightarrow P_2 \\ p(x) &\longmapsto p'(x) \end{aligned}$$

(c) $$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (x + 2y + 3z, x + z, x + y + 2z) \end{aligned}$$

(These are the linear transformations from Exercise C105.)

4.2 Kernel of a linear transformation

You have seen how to find the image set of a linear transformation $t : V \longrightarrow W$. Now suppose that $\mathbf{w}$ belongs to the image set of t. How can we find all the vectors in V that map to $\mathbf{w}$?

We begin by looking at the case when $\mathbf{w}$ is the zero vector. We know that $t(\mathbf{0}) = \mathbf{0}$, but it is possible that there are also some non-zero vectors in V that are mapped to $\mathbf{0}$.

For example, let t be the linear transformation

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (x, y, 0). \end{aligned}$$

Then $t(x, y, z) = \mathbf{0}$ if and only if $(x, y, 0) = (0, 0, 0)$; that is, if and only if $x = 0$ and $y = 0$.

Thus the set of vectors that are mapped to $\mathbf{0}$ is the whole of the z-axis. This set is a one-dimensional subspace of the domain $\mathbb{R}^3$. We call this set the *kernel* of t.

The first use of *kernel* in the context of algebra was by the Russian mathematician Lev Semyonovich Pontryagin (1908–1988) in 1931. Pontryagin, who lost his eyesight in an accident when he was fourteen, was one of the leading Russian mathematicians of the twentieth century. He made fundamental contributions to algebra, topology, and dynamical systems.

Pontryagin's choice of the term *kernel* appears unrelated to its use in other areas of mathematics (integral equations, Fourier analysis).

Lev Semyonovich Pontryagin

Definition

The **kernel** of a linear transformation $t : V \longrightarrow W$ is the set

$$\operatorname{Ker} t = \{\mathbf{v} \in V : t(\mathbf{v}) = \mathbf{0}\}.$$

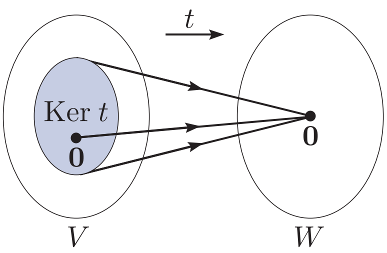


Figure 29 The kernel maps to the zero vector

Figure 29 illustrates that the kernel is the set of vectors of V mapping to the zero vector of W.

Exercise C107

Give a geometric description of the kernel of each of the following linear transformations. In each case, state whether the kernel is a subspace of the domain.

(a) $t : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$
$(x, y, z) \longmapsto (x, 0)$

(b) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (x, x)$

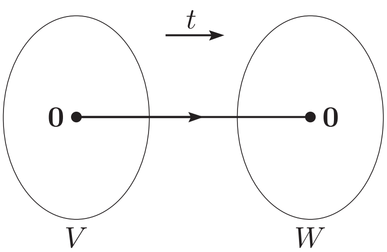


Figure 30 The vector $\mathbf{0}$ is in the kernel

For each of the linear transformations in Exercise C107, the kernel is a subspace of the domain. This is true for all linear transformations.

Theorem C50

Let $t : V \longrightarrow W$ be a linear transformation. Then $\operatorname{Ker} t$ is a subspace of the domain V.

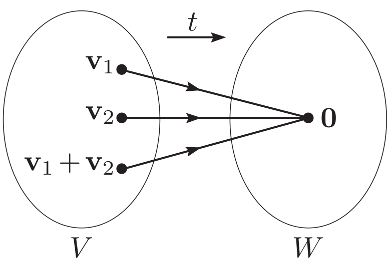


Figure 31 The kernel is closed under vector addition

Proof We use Strategy C10 in Unit C2.

First we show that $\mathbf{0} \in \operatorname{Ker} t$.

Since t is a linear transformation, $t(\mathbf{0}) = \mathbf{0}$, so $\mathbf{0} \in \operatorname{Ker} t$.

This is illustrated in Figure 30.

Next we show that $\operatorname{Ker} t$ is closed under vector addition.

Let $\mathbf{v}_1, \mathbf{v}_2 \in \operatorname{Ker} t$. Since t is a linear transformation,

$$t(\mathbf{v}_1 + \mathbf{v}_2) = t(\mathbf{v}_1) + t(\mathbf{v}_2) = \mathbf{0} + \mathbf{0} = \mathbf{0},$$

so $\mathbf{v}_1 + \mathbf{v}_2 \in \operatorname{Ker} t$, as required.

This is illustrated in Figure 31.

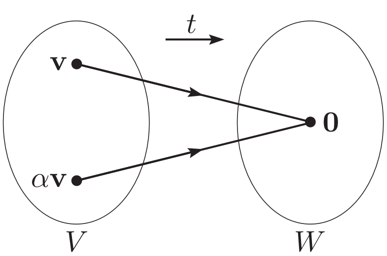


Figure 32 The kernel is closed under scalar multiplication

Finally, we show that $\operatorname{Ker} t$ is closed under scalar multiplication.

Let $\mathbf{v} \in \operatorname{Ker} t$ and $\alpha \in \mathbb{R}$. Since t is a linear transformation,

$$t(\alpha\mathbf{v}) = \alpha\, t(\mathbf{v}) = \alpha\mathbf{0} = \mathbf{0},$$

so $\alpha\mathbf{v} \in \operatorname{Ker} t$.

This is illustrated in Figure 32.

Thus $\operatorname{Ker} t$ is a subspace of V. ■

When finding the kernel of a linear transformation, we often need to solve a system of linear equations; this sometimes involves using Gauss–Jordan elimination as in Unit C1.

Worked Exercise C60

Find the kernel and the dimension of the kernel of the linear transformation

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ (x, y, z) &\longmapsto (x + 2y + 3z, 4x + y - 2z). \end{aligned}$$

Solution

The kernel is the set of vectors (x, y, z) in $\mathbb{R}^3$ that satisfy

$$t(x, y, z) = \mathbf{0},$$

that is,

$$(x + 2y + 3z, 4x + y - 2z) = (0, 0).$$

Equating coordinates, we obtain the system

$$\begin{aligned} x + 2y + 3z &= 0 \\ 4x + y - 2z &= 0. \end{aligned}$$

To solve this system we row-reduce the augmented matrix.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \end{array} \quad \left(\begin{array}{rrr|r} 1 & 2 & 3 & 0 \\ 4 & 1 & -2 & 0 \end{array}\right) \quad \begin{array}{r} 6 \\ 3 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 4\mathbf{r}_1 \end{array} \quad \left(\begin{array}{rrr|r} 1 & 2 & 3 & 0 \\ 0 & -7 & -14 & 0 \end{array}\right) \quad \begin{array}{r} 6 \\ -21 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to -\frac{1}{7}\mathbf{r}_2 \end{array} \quad \left(\begin{array}{rrr|r} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \quad \begin{array}{r} 6 \\ 3 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_2 \\ \\ \end{array} \quad \left(\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \quad \begin{array}{r} 0 \\ 3 \end{array}$$

The augmented matrix is in row-reduced form and we have

$$\begin{aligned} x \quad - \quad z &= 0 \\ y + 2z &= 0. \end{aligned}$$

Assigning the parameter k to the unknown z, we obtain

$$x = k, \quad y = -2k, \quad z = k.$$

So the kernel of t is

$$\operatorname{Ker} t = \{(k, -2k, k) : k \in \mathbb{R}\};$$

that is, $\operatorname{Ker} t$ is the line through $(0, 0, 0)$ and $(1, -2, 1)$.

Thus $\operatorname{Ker} t$ is a one-dimensional subspace of the domain $\mathbb{R}^3$.

Exercise C108

For each of the following linear transformations t, find the kernel of t and the dimension of the kernel.

(a) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (x, 2x + y)$

(b) $t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$
$(x, y, z) \longmapsto (x + 2y + 3z, x + z, x + y + 2z)$

We now look at examples involving vector spaces of polynomials.

Worked Exercise C61

Find the kernel of the linear transformation

$$\begin{aligned} t : P_3 &\longrightarrow P_3 \\ p(x) &\longmapsto p(x) + p(2). \end{aligned}$$

Solution

Let $p(x) = a + bx + cx^2$ be a polynomial in P_3. Then

$$\begin{aligned} t(p(x)) &= a + bx + cx^2 + a + 2b + 4c \\ &= 2a + 2b + 4c + bx + cx^2. \end{aligned}$$

The kernel of t is the set of polynomials in P_3 that satisfy $t(p(x)) = \mathbf{0}$; that is,

$$2a + 2b + 4c + bx + cx^2 = 0, \quad \text{for all } x \in \mathbb{R}.$$

Equating coefficients, we obtain the system

$$\begin{aligned} 2a + 2b + 4c &= 0 \\ b \qquad &= 0 \\ c &= 0. \end{aligned}$$

Substituting $b = 0$ and $c = 0$ into the first equation gives $a = 0$. So the only solution is $a = 0$, $b = 0$ and $c = 0$.

Thus the only polynomial in the kernel of t is the zero polynomial $p(x) = 0$; that is,

$$\operatorname{Ker} t = \{\mathbf{0}\}.$$

The kernel comprises just the zero vector so it has dimension 0.

Exercise C109

Find the kernel and dimension of the kernel of the linear transformation

$$\begin{aligned} t : P_3 &\longrightarrow P_2 \\ p(x) &\longmapsto p'(x). \end{aligned}$$

For a given linear transformation $t : V \longrightarrow W$, we know how to find all the vectors in V that map to $\mathbf{0}$ in W. Now suppose that $\mathbf{b}$ ($\neq \mathbf{0}$) is some particular vector in W. How do we find all the vectors in V that map to $\mathbf{b}$? This is illustrated in Figure 33. There is a close relationship between the vectors that map to $\mathbf{b}$ and those that map to $\mathbf{0}$: if we know *one* vector $\mathbf{a}$ in V that maps to $\mathbf{b}$, that is, $t(\mathbf{a}) = \mathbf{b}$, then *every* vector $\mathbf{x}$ in V that maps to $\mathbf{b}$ may be written in the form $\mathbf{x} = \mathbf{a} + \mathbf{k}$, where $t(\mathbf{k}) = \mathbf{0}$, that is, $\mathbf{k} \in \operatorname{Ker} t$. We state this powerful result formally in the following theorem; the proof is short and constructive.

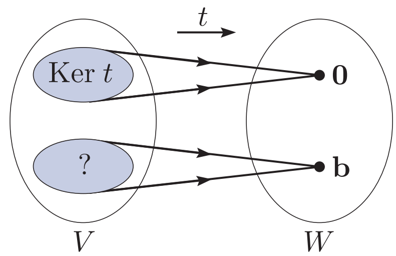


Figure 33 The vectors of V mapping to $\mathbf{b}$

Theorem C51 Solution Set Theorem

Let $t : V \longrightarrow W$ be a linear transformation. Let $\mathbf{b} \in W$ and let $\mathbf{a}$ be one vector in V that maps to $\mathbf{b}$, that is, $t(\mathbf{a}) = \mathbf{b}$. Then the solution set of the equation $t(\mathbf{x}) = \mathbf{b}$ is

$$\{\mathbf{x} : \mathbf{x} = \mathbf{a} + \mathbf{k} \text{ for some } \mathbf{k} \in \operatorname{Ker} t\}.$$

Proof The proof is in two parts. We first show that the given set is a subset of the solution set.

First we show that each vector $\mathbf{x}$ of the given form is a solution of $t(\mathbf{x}) = \mathbf{b}$. Let $\mathbf{x} = \mathbf{a} + \mathbf{k}$, where $\mathbf{k} \in \operatorname{Ker} t$. Then

$$t(\mathbf{x}) = t(\mathbf{a} + \mathbf{k}) = t(\mathbf{a}) + t(\mathbf{k}) = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

We now show that the solution set is a subset of the given set.

Conversely, we show that each vector $\mathbf{x}$ in the solution set has the given form. Let $t(\mathbf{x}) = \mathbf{b}$, where $\mathbf{x} \in V$. Then

$$t(\mathbf{x} - \mathbf{a}) = t(\mathbf{x}) - t(\mathbf{a}) = \mathbf{b} - \mathbf{b} = \mathbf{0},$$

so $\mathbf{x} - \mathbf{a} \in \operatorname{Ker} t$; that is, $\mathbf{x} = \mathbf{a} + \mathbf{k}$, for some $\mathbf{k} \in \operatorname{Ker} t$. ■

Finally, we recall that a linear transformation $t : V \longrightarrow W$ is one-to-one if and only if no two elements in V have the same image. Thus we have the following result.

Proposition C52

A linear transformation t is one-to-one if and only if $\operatorname{Ker} t = \{\mathbf{0}\}$.

Exercise C110

Which of the following linear transformations are one-to-one?

(a) $$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x, 2x + y) \end{aligned}$$

(b) $$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (x + 2y + 3z, x + z, x + y + 2z) \end{aligned}$$

(c) $$\begin{aligned} t : P_3 &\longrightarrow P_2 \\ p(x) &\longmapsto p'(x) \end{aligned}$$

(You found the kernels of these in Exercises C108 and C109.)

4.3 Dimension Theorem

You have seen that a linear transformation $t : V \longrightarrow W$ has two particular subspaces associated with it: $\operatorname{Ker} t$ in the domain V and $\operatorname{Im} t$ in the codomain W, as show in Figure 34.

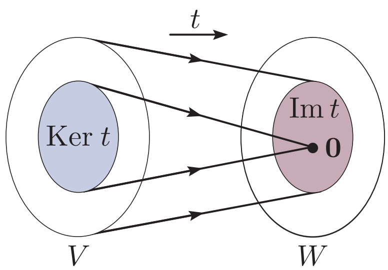


Figure 34 The subspaces $\operatorname{Ker} t$ and $\operatorname{Im} t$

There is a remarkable connection between the dimensions of these two subspaces and the dimension of the domain V.

Let t be the linear transformation

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (x, y, 0). \end{aligned}$$

You have seen that for this linear transformation:

- the image set of t is the (x, y)-plane, so $\dim(\operatorname{Im} t) = 2$
- the kernel of t is the z-axis, so $\dim(\operatorname{Ker} t) = 1$.

Thus

$$\dim(\operatorname{Im} t) + \dim(\operatorname{Ker} t) = 2 + 1 = 3,$$

which is the dimension of the domain $\mathbb{R}^3$.

Now let t be the linear transformation

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ (x, y, z) &\longmapsto (x + 2y + 3z, 4x + y - 2z). \end{aligned}$$

You have seen that for this linear transformation:

- the image set of t is the whole of $\mathbb{R}^2$, so $\dim(\operatorname{Im} t) = 2$
- the kernel of t is the line through $(0,0,0)$ and $(1,-2,1)$, so $\dim(\operatorname{Ker} t) = 1$.

Thus

$$\dim(\operatorname{Im} t) + \dim(\operatorname{Ker} t) = 2 + 1 = 3,$$

which is the dimension of the domain $\mathbb{R}^3$.

Exercise C111

For each of the following linear transformations t, calculate

$$\dim(\operatorname{Im} t) + \dim(\operatorname{Ker} t)$$

and compare your answer with the dimension of the domain of t.

(a) $\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x, 2x+y) \end{aligned}$

(b) $\begin{aligned} t : P_3 &\longrightarrow P_2 \\ p(x) &\longmapsto p'(x) \end{aligned}$

(c) $\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x,y,z) &\longmapsto (x+2y+3z, x+z, x+y+2z) \end{aligned}$

(You found the bases and dimensions of the image sets in Exercise C105, and the kernels and dimensions of the kernels in Exercises C108 and C109.)

For each of the linear transformations in Exercise C111, the dimension of the image set plus the dimension of the kernel is equal to the dimension of the domain. This relationship holds for all linear transformations. We state this result in the next theorem; if you are short of time you should skim through this proof and come back to it when time permits.

Theorem C53 Dimension Theorem

Let $t : V \longrightarrow W$ be a linear transformation. Then

$$\dim(\operatorname{Im} t) + \dim(\operatorname{Ker} t) = \dim V.$$

Proof Let $\dim V = n$ and $\dim(\operatorname{Ker} t) = k$.

We show that $\dim(\operatorname{Im} t) = n - k$.

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ be a basis for $\operatorname{Ker} t$. We can extend this basis, by Theorem C26 in Unit C2, to give a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for V. We prove that

$$F = \{t(\mathbf{e}_{k+1}), \ldots, t(\mathbf{e}_n)\}$$

is a basis for $\operatorname{Im} t$, which shows that $\dim(\operatorname{Im} t) = n - k$.

A diagram helps here: see Figure 35.

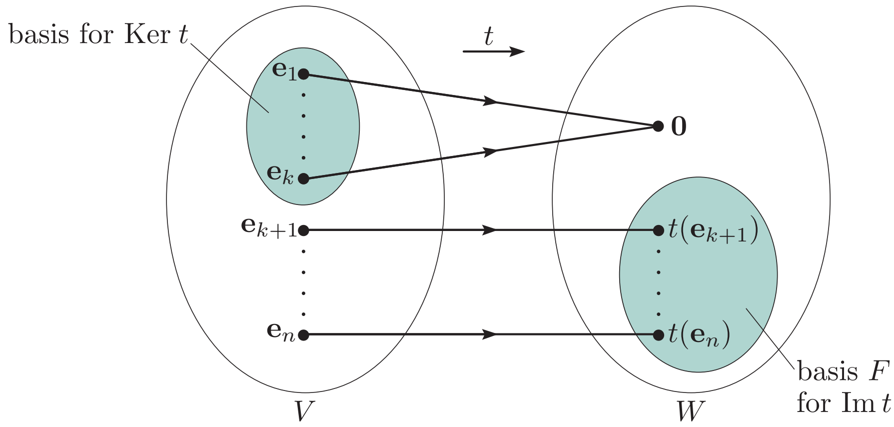


Figure 35 Bases for Ker t in V and Im t in W

To show that F is a basis for Im t, we use Strategy C8 in Unit C2.

First we prove that F spans Im t. We know from Subsection 4.1 that $\{t(\mathbf{e}_1), \ldots, t(\mathbf{e}_n)\}$ spans Im t. Since $\mathbf{e}_1, \ldots, \mathbf{e}_k$ belong to Ker t, we know that

$$t(\mathbf{e}_1) = t(\mathbf{e}_2) = \cdots = t(\mathbf{e}_k) = \mathbf{0},$$

so the span of $\{t(\mathbf{e}_1), \ldots, t(\mathbf{e}_n)\}$ is equal to the span of $\{t(\mathbf{e}_{k+1}), \ldots, t(\mathbf{e}_n)\}$. Thus F spans Im t.

Next we show that F is a linearly independent set. We must show that if

$$\alpha_{k+1}t(\mathbf{e}_{k+1}) + \alpha_{k+2}t(\mathbf{e}_{k+2}) + \cdots + \alpha_n t(\mathbf{e}_n) = \mathbf{0},$$

then

$$\alpha_{k+1} = \alpha_{k+2} = \cdots = \alpha_n = 0.$$

Since t is a linear transformation, we have

$$\alpha_{k+1}t(\mathbf{e}_{k+1}) + \cdots + \alpha_n t(\mathbf{e}_n) = t(\alpha_{k+1}\mathbf{e}_{k+1} + \cdots + \alpha_n\mathbf{e}_n).$$

So if

$$\alpha_{k+1}t(\mathbf{e}_{k+1}) + \cdots + \alpha_n t(\mathbf{e}_n) = \mathbf{0},$$

then

$$t(\alpha_{k+1}\mathbf{e}_{k+1} + \cdots + \alpha_n\mathbf{e}_n) = \mathbf{0}.$$

Thus

$$\alpha_{k+1}\mathbf{e}_{k+1} + \cdots + \alpha_n\mathbf{e}_n \in \operatorname{Ker} t.$$

Since $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ is a basis for Ker t, there exist real numbers $\alpha_1, \ldots, \alpha_k$ such that

$$\alpha_{k+1}\mathbf{e}_{k+1} + \cdots + \alpha_n\mathbf{e}_n = \alpha_1\mathbf{e}_1 + \cdots + \alpha_k\mathbf{e}_k,$$

so

$$\alpha_1\mathbf{e}_1 + \cdots + \alpha_k\mathbf{e}_k - \alpha_{k+1}\mathbf{e}_{k+1} - \cdots - \alpha_n\mathbf{e}_n = \mathbf{0}.$$

Since $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis for V and so is linearly independent, it follows that

$$\alpha_1 = \cdots = \alpha_k = -\alpha_{k+1} = \cdots = -\alpha_n = 0.$$

Thus

$$\alpha_{k+1} = \alpha_{k+2} = \cdots = \alpha_n = 0,$$

as required.

Thus F is a basis for $\operatorname{Im} t$, so $\dim(\operatorname{Im} t) + \dim(\operatorname{Ker} t) = \dim V$. ■

The Dimension Theorem is an important result and has several applications. For example, using the Dimension Theorem we can obtain information on whether a linear transformation $t : V \longrightarrow W$ is one-to-one and/or onto.

Propositions C49 and C52 state that:

- t is onto if and only if $\dim(\operatorname{Im} t) = \dim W$
- t is one-to-one if and only if $\operatorname{Ker} t = \{\mathbf{0}\}$.

Suppose that $t : V \longrightarrow W$ is a linear transformation from the n-dimensional vector space V to the m-dimensional vector space W, as illustrated in Figure 36.

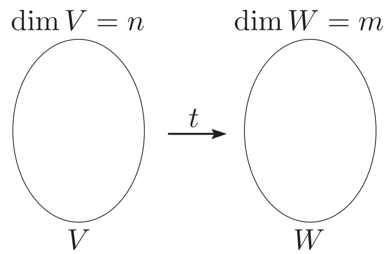


Figure 36 A linear transformation from V to W

We consider the three cases: $n > m$, $n < m$ and $n = m$.

Case (a): $n > m$

Since the image set of t is a subspace of W, we have $\dim(\operatorname{Im} t) \leq m$. It follows from the Dimension Theorem that

$$\dim(\operatorname{Ker} t) = \dim V - \dim(\operatorname{Im} t) \geq n - m > 0.$$

Thus $\operatorname{Ker} t \neq \{\mathbf{0}\}$, so t is not one-to-one, as illustrated in Figure 37.

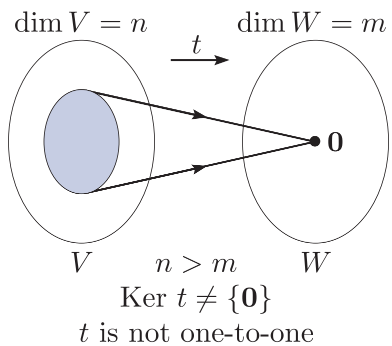


Figure 37 The case $\dim V > \dim W$

For example, the linear transformation

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ (x, y, z) &\longmapsto (2x + y, x + z) \end{aligned}$$

is not one-to-one, since the dimension of the codomain (which is 2) is less than the dimension of the domain (which is 3).

This linear transformation is onto because $\dim(\operatorname{Im} t) = 2 = \dim \mathbb{R}^2$. However, in general, a linear transformation with $n > m$ may or may not be onto.

Case (b): $n < m$

By the Dimension Theorem,

$$\dim(\operatorname{Im} t) = \dim V - \dim(\operatorname{Ker} t) \leq n < m.$$

Thus $\operatorname{Im} t$ is not the whole of the m-dimensional vector space W, so t is not onto, as illustrated in Figure 38.

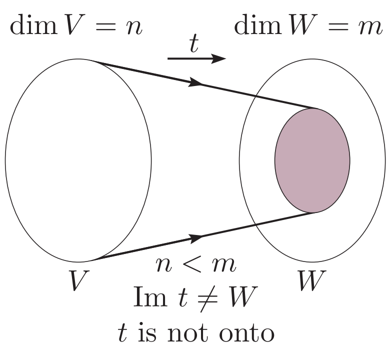


Figure 38 The case $\dim V < \dim W$

For example, the linear transformation

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ (x, y) &\longmapsto (2x, x + y, y) \end{aligned}$$

is not onto, since the dimension of the codomain (which is 3) is greater than the dimension of the domain (which is 2).

This linear transformation is one-to-one because $\dim(\operatorname{Im} t) = 2 = \dim \mathbb{R}^2$. However, in general, a linear transformation with $n < m$ may or may not be one-to-one.

Case (c): $n = m$

By the Dimension Theorem,

$$\dim(\operatorname{Im} t) + \dim(\operatorname{Ker} t) = \dim V = n = m.$$

There are two possibilities:

- $\dim(\operatorname{Ker} t) = 0$ and $\dim(\operatorname{Im} t) = n = m$
- $\dim(\operatorname{Ker} t) > 0$ and $\dim(\operatorname{Im} t) < m$.

If $\dim(\operatorname{Ker} t) = 0$ and $\dim(\operatorname{Im} t) = n = m$, then

$$\operatorname{Ker} t = \{\mathbf{0}\} \quad \text{and} \quad \operatorname{Im} t = W.$$

Thus t is both one-to-one and onto, as illustrated in Figure 39.

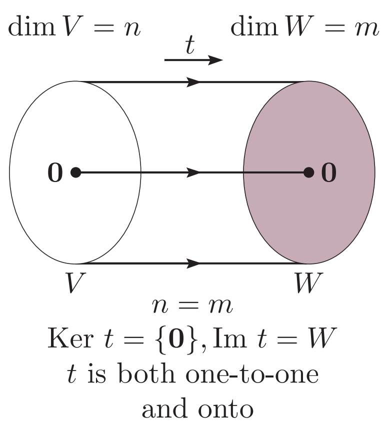


Figure 39 The case $\dim V = \dim W$ and $\operatorname{Ker} t = \{\mathbf{0}\}$

For example, consider the linear transformation from Exercise C105(a),

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x, 2x + y). \end{aligned}$$

Here the domain and codomain both have dimension 2, $\dim(\operatorname{Ker} t) = 0$ and $\dim(\operatorname{Im} t) = 2$. The latter is equal to the dimension of the codomain, so t is both one-to-one and onto.

If, on the other hand, $\dim(\operatorname{Ker} t) > 0$ and $\dim(\operatorname{Im} t) < m$, then

$$\operatorname{Ker} t \neq \{\mathbf{0}\} \quad \text{and} \quad \operatorname{Im} t \text{ is not the whole of } W.$$

Thus t is neither one-to-one nor onto, as illustrated in Figure 40.

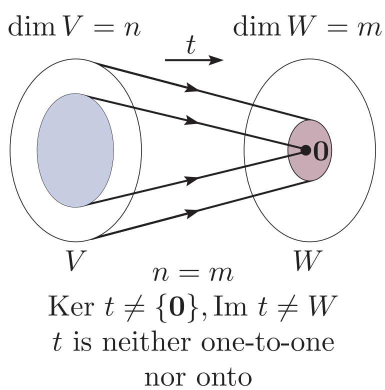


Figure 40 The case $\dim V = \dim W$ and $\operatorname{Ker} t \neq \{\mathbf{0}\}$

For example, consider the linear transformation from Exercise C105(c),

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (x + 2y + 3z, x + z, x + y + 2z). \end{aligned}$$

Here the domain and codomain both have dimension 3, $\dim(\operatorname{Ker} t) = 1$ and $\dim(\operatorname{Im} t) = 2$. The latter is less than the dimension of the codomain of t, and thus t is neither one-to-one nor onto.

We summarise these findings in the following theorem.

Theorem C54

Let $t : V \longrightarrow W$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W.

(a) If $n > m$, then t is not one-to-one: $\operatorname{Ker} t \neq \{\mathbf{0}\}$.

(b) If $n < m$, then t is not onto: $\operatorname{Im} t \neq W$.

(c) If $n = m$, then

- *either* t is both one-to-one and onto:

  $\operatorname{Ker} t = \{\mathbf{0}\}$ and $\operatorname{Im} t = W$

- *or* t is neither one-to-one nor onto:

  $\operatorname{Ker} t \neq \{\mathbf{0}\}$ and $\operatorname{Im} t \neq W$.

Exercise C112

What can we deduce from Theorem C54 about the following linear transformations?

(a) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$
$(x, y) \longmapsto (x, y, x + y)$

(b) $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x, y) \longmapsto (3x, 4x + y)$

(c) $t : P_3 \longrightarrow P_2$
$p(x) \longmapsto p'(x)$

Systems of linear equations

You will now see how we can use linear transformations to obtain information on the number of solutions of a system of linear equations.

Suppose that we want to know how many solutions there are to the following system of three linear equations in three unknowns:

$$\begin{aligned} 2x + 3y + 4z &= 7\\ x + 5y + 6z &= 4\\ 3x + 2y + 5z &= 1. \end{aligned}$$

This system can be written in matrix form as

$$\begin{pmatrix} 2 & 3 & 4\\ 1 & 5 & 6\\ 3 & 2 & 5 \end{pmatrix} \begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} 7\\ 4\\ 1 \end{pmatrix}.$$

Now let t be the linear transformation with the matrix representation

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3\\ \begin{pmatrix} x\\ y\\ z \end{pmatrix} &\longmapsto \begin{pmatrix} 2 & 3 & 4\\ 1 & 5 & 6\\ 3 & 2 & 5 \end{pmatrix} \begin{pmatrix} x\\ y\\ z \end{pmatrix}. \end{aligned}$$

We see that (x, y, z) is a solution of the system of equations precisely when $t(x, y, z) = (7, 4, 1)$.

Thus the number of solutions of the system of equations is the same as the number of vectors in $\mathbb{R}^3$ that map to the vector $(7, 4, 1)$ under t.

In general, suppose that we want to know how many solutions there are to the system of m linear equations in n unknowns with the matrix equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}.$$

Let t be the linear transformation with the matrix representation

$$\begin{aligned} t : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ \mathbf{x} &\longmapsto \mathbf{A}\mathbf{x}. \end{aligned}$$

Then the number of solutions of the system of equations is the same as the number of vectors that map to $\mathbf{b}$ under t.

Suppose $\mathbf{b} \in \operatorname{Im} t$. Then there is some vector $\mathbf{a} \in \mathbb{R}^n$ such that $t(\mathbf{a}) = \mathbf{b}$. Then, using the Solution Set Theorem (Theorem C51), the solution set to the system of equations is

$$\{\mathbf{x} : \mathbf{x} = \mathbf{a} + \mathbf{k} \text{ for some } \mathbf{k} \in \operatorname{Ker} t\}.$$

Now $\operatorname{Ker} t$ is a subspace of $\mathbb{R}^n$, by Theorem C50. A subspace of $\mathbb{R}^n$ of dimension 0 comprises just the zero vector. A subspace of $\mathbb{R}^n$ of dimension greater than 0 comprises infinitely many vectors since it is a line, a plane or a higher-dimensional space. So $\operatorname{Ker} t$ contains either just the zero vector or infinitely many vectors.

It follows that there are three possibilities for the number of solutions:

- if $\mathbf{b} \in \operatorname{Im} t$ and $\operatorname{Ker} t = \{\mathbf{0}\}$, then there is exactly *one* solution
- if $\mathbf{b} \in \operatorname{Im} t$ and $\operatorname{Ker} t \neq \{\mathbf{0}\}$, then there are *infinitely many* solutions
- if $\mathbf{b} \notin \operatorname{Im} t$, then there are *no* solutions.

Thus a system of linear equations has no solutions, or one solution, or infinitely many solutions. This result was stated without proof in Subsection 1.2 of Unit C1.

Exercise C113

How many solutions are there to the following system of three linear equations in three unknowns?

$$\begin{aligned} x + 2y + 3z &= 1 \\ x \qquad + \ \ z &= 1 \\ x + \ \ y + 2z &= 1 \end{aligned}$$

Use your solutions to Exercises C105(c) and C108(b).

By considering the linear transformation

$$\begin{aligned} t : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ \mathbf{x} &\longmapsto \mathbf{A}\mathbf{x}, \end{aligned}$$

we can show that the number of solutions of the system $\mathbf{Ax} = \mathbf{b}$ of m linear equations in n unknowns depends on the values of m and n. We consider three cases: $n > m$, $n < m$ and $n = m$.

Case (a): $n > m$

It follows from Theorem C54 that $\operatorname{Ker} t \neq \{\mathbf{0}\}$. Thus the equation $\mathbf{Ax} = \mathbf{b}$ has either no solution (if $\mathbf{b} \notin \operatorname{Im} t$) or infinitely many solutions (if $\mathbf{b} \in \operatorname{Im} t$). For example, the system

$$\begin{aligned} 2x + y + z &= a \\ 4x + 2y + 2z &= b, \end{aligned}$$

of two equations in three unknowns has either no solution or infinitely many solutions, depending on the values of a and b. For example, the system has no solution when $a = 3$ and $b = 4$, and infinitely many solutions when $a = 2$ and $b = 4$.

Case (b): $n < m$

It follows from Theorem C54 that $\operatorname{Im} t \neq \mathbb{R}^m$. Thus there is some $\mathbf{b}$ for which the equation $\mathbf{Ax} = \mathbf{b}$ has no solutions. For example, there are some values of a, b and c for which the system

$$\begin{aligned} 2x + y &= a \\ x + 3y &= b \\ 4x + y &= c, \end{aligned}$$

of three equations in two unknowns has no solutions. For example, the system has no solutions when $a = 3$, $b = 4$ and $c = 2$.

Case (c): $n = m$

It follows from Theorem C54 that there are two possibilities.

If $\operatorname{Ker} t = \{\mathbf{0}\}$ and $\operatorname{Im} t = \mathbb{R}^m$, then the equation $\mathbf{Ax} = \mathbf{b}$ has exactly one solution for each $\mathbf{b}$. For example, the system

$$\begin{aligned} x + y &= a \\ y &= b, \end{aligned}$$

of two equations in two unknowns has exactly one solution, namely $(x, y) = (a - b, b)$, for each pair of values (a, b).

If $\operatorname{Ker} t \neq \{\mathbf{0}\}$ and $\operatorname{Im} t \neq \mathbb{R}^m$, then there exist vectors $\mathbf{b}$ for which the equation $\mathbf{Ax} = \mathbf{b}$ has no solutions, and for all other $\mathbf{b}$, the equation $\mathbf{Ax} = \mathbf{b}$ has infinitely many solutions. Consider the system

$$\begin{aligned} x + 2y &= a \\ 2x + 4y &= b, \end{aligned}$$

of two equations in two unknowns. Since $2x + 4y = 2(x + 2y)$, these equations have no solution when $b \neq 2a$. When $b = 2a$, however, putting $y = k$ gives $(x, y) = (a - 2k, k)$, where $k \in \mathbb{R}$, as a solution of the equations; thus there are infinitely many solutions.

We summarise these results below.

Theorem C55

Let $\mathbf{Ax} = \mathbf{b}$ be a system of m linear equations in n unknowns.

(a) If $n > m$, then $\mathbf{Ax} = \mathbf{b}$ has either no solution or infinitely many solutions.

(b) If $n < m$, then there is some $\mathbf{b}$ for which $\mathbf{Ax} = \mathbf{b}$ has no solutions.

(c) If $n = m$, then:

- *either* $\mathbf{Ax} = \mathbf{b}$ has exactly one solution for each $\mathbf{b}$
- *or* there are some $\mathbf{b}$ for which $\mathbf{Ax} = \mathbf{b}$ has no solutions; for all other $\mathbf{b}$, $\mathbf{Ax} = \mathbf{b}$ has infinitely many solutions.

Exercise C114

What can you deduce from Theorem C55 about the number of solutions of each of the following systems of linear equations?

(a) $$\begin{aligned} 3x + y + z &= 1 \\ 4x + 2y + 4z &= 3 \end{aligned}$$

(b) $$\begin{aligned} 3x + y + z &= a \\ 4x + 2y + 4z &= b \\ 5x + y + 6z &= c \end{aligned}$$

Summary

In this unit you have seen that linear transformations are functions between vector spaces that preserve linear combinations of vectors, and that for finite-dimensional vector spaces they are precisely the functions that have matrix representations. Using properties of matrices you have investigated invertible linear transformations. You have seen that finite-dimensional vector spaces are isomorphic if and only if their dimensions are equal, and hence that all vector spaces of dimension n are isomorphic to $\mathbb{R}^n$. You have met the Dimension Theorem, the important result that the sum of dimensions of the image set and kernel are equal to the dimension of the domain. In addition, you have seen that linear transformations can be used to prove that matrix multiplication is associative and to help determine the number of solutions of a system of linear equations.

Learning outcomes

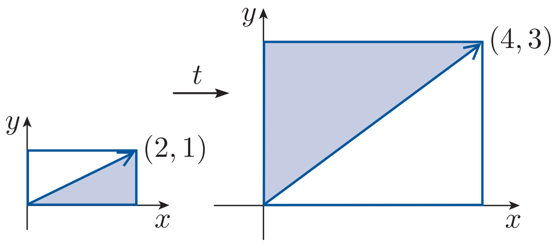
After working through this unit, you should be able to:

- explain what is meant by a *linear transformation* and understand that linear transformations preserve the zero vector and linear combinations of vectors
- recognise simple linear transformations of the plane
- determine whether or not a given function is a linear transformation
- understand that the *matrix representation* of a linear transformation $t : V \longrightarrow W$ depends on the bases used for V and W
- find the matrix representation, with respect to given bases, of a linear transformation between finite-dimensional vector spaces
- understand the relationship between matrices and linear transformations
- use the matrix representations of two given linear transformations s and t to find a matrix representation of the composite function $s \circ t$
- determine whether a given linear transformation is invertible and, if it is, find its inverse
- understand that each n-dimensional vector space is isomorphic to $\mathbb{R}^n$
- explain the meaning of the terms *image set* and *kernel* of a linear transformation
- find a basis for the image set of a given linear transformation and find the kernel of a given linear transformation
- understand the relationship between the dimension of the image set, the dimension of the kernel and the dimension of the domain of a linear transformation
- understand that the number of solutions of a system of m linear equations in n unknowns depends on the values of m and n.

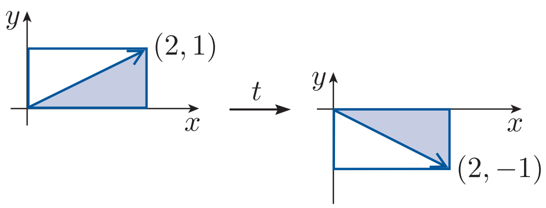
Solutions to exercises

Solution to Exercise C82

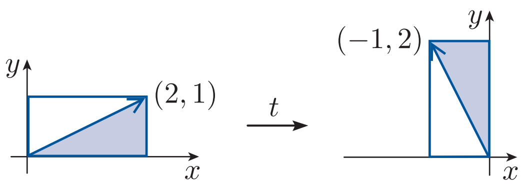
(a) This is a $(2,3)$-scaling.



(b) This is q_0, a reflection in the x-axis; it is also a $(1,-1)$-scaling.



(c) This is $r_{\pi/2}$, a rotation through an angle $\pi/2$.



Solution to Exercise C83

(a) First $t(\mathbf{0}) = \mathbf{0}$, so t may be a linear transformation.

Next we check whether t satisfies LT1:

$$t(\mathbf{v}_1 + \mathbf{v}_2) = t(\mathbf{v}_1) + t(\mathbf{v}_2), \quad \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2.$$

In $\mathbb{R}^2$, let $\mathbf{v}_1 = (x_1, y_1)$ and $\mathbf{v}_2 = (x_2, y_2)$. Then

$$\begin{aligned} t(\mathbf{v}_1 + \mathbf{v}_2) &= t(x_1 + x_2, y_1 + y_2) \\ &= (x_1 + x_2 + 3(y_1 + y_2), y_1 + y_2) \\ &= (x_1 + x_2 + 3y_1 + 3y_2, y_1 + y_2) \end{aligned}$$

and

$$\begin{aligned} t(\mathbf{v}_1) + t(\mathbf{v}_2) &= (x_1 + 3y_1, y_1) + (x_2 + 3y_2, y_2) \\ &= (x_1 + x_2 + 3y_1 + 3y_2, y_1 + y_2). \end{aligned}$$

These expressions are equal, so LT1 is satisfied.

Finally, we check whether t satisfies LT2:

$$t(\alpha\mathbf{v}) = \alpha\, t(\mathbf{v}), \quad \text{for all } \mathbf{v} \in \mathbb{R}^2,\ \alpha \in \mathbb{R}.$$

Let $\mathbf{v} = (x, y)$ be a vector in $\mathbb{R}^2$ and let $\alpha \in \mathbb{R}$. Then

$$t(\alpha\mathbf{v}) = t(\alpha x, \alpha y) = (\alpha x + 3\alpha y, \alpha y)$$

and

$$\alpha\, t(\mathbf{v}) = \alpha(x + 3y, y) = (\alpha x + 3\alpha y, \alpha y).$$

These expressions are equal, so LT2 is satisfied.

Since both LT1 and LT2 are satisfied, t is a linear transformation.

(b) Since $t(\mathbf{0}) = t(0,0) = (2,1) \neq \mathbf{0}$, it follows from Strategy C14 that t is not a linear transformation.

Solution to Exercise C84

We use Strategy C14.

(a) First $t(\mathbf{0}) = \mathbf{0}$, so t may be a linear transformation.

Next we check whether t satisfies LT1:

$$t(\mathbf{v}_1 + \mathbf{v}_2) = t(\mathbf{v}_1) + t(\mathbf{v}_2), \quad \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2.$$

In $\mathbb{R}^2$, let $\mathbf{v}_1 = (x_1, y_1)$ and $\mathbf{v}_2 = (x_2, y_2)$. Then

$$\begin{aligned} t(\mathbf{v}_1 + \mathbf{v}_2) &= t(x_1 + x_2, y_1 + y_2) \\ &= (x_1 + x_2, y_1 + y_2, x_1 + x_2, y_1 + y_2) \end{aligned}$$

and

$$\begin{aligned} t(\mathbf{v}_1) + t(\mathbf{v}_2) &= (x_1, y_1, x_1, y_1) + (x_2, y_2, x_2, y_2) \\ &= (x_1 + x_2, y_1 + y_2, x_1 + x_2, y_1 + y_2). \end{aligned}$$

These expressions are equal, so LT1 is satisfied.

Finally, we check whether t satisfies LT2:

$$t(\alpha\mathbf{v}) = \alpha\, t(\mathbf{v}), \quad \text{for all } \mathbf{v} \in \mathbb{R}^2,\ \alpha \in \mathbb{R}.$$

Let $\mathbf{v} = (x, y)$ be a vector in $\mathbb{R}^2$ and let $\alpha \in \mathbb{R}$. Then

$$t(\alpha\mathbf{v}) = t(\alpha x, \alpha y) = (\alpha x, \alpha y, \alpha x, \alpha y)$$

and

$$\alpha\, t(\mathbf{v}) = \alpha(x, y, x, y) = (\alpha x, \alpha y, \alpha x, \alpha y).$$

These expressions are equal, so LT2 is satisfied.

Since both LT1 and LT2 are satisfied, t is a linear transformation.

(b) First $t(\mathbf{0}) = \mathbf{0}$, so t may be a linear transformation.

Next we check whether t satisfies LT1:

$$t(\mathbf{v}_1 + \mathbf{v}_2) = t(\mathbf{v}_1) + t(\mathbf{v}_2), \quad \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3.$$

In $\mathbb{R}^3$, let $\mathbf{v}_1 = (x_1, y_1, z_1)$ and $\mathbf{v}_2 = (x_2, y_2, z_2)$. Then

$$\begin{aligned} t(\mathbf{v}_1 + \mathbf{v}_2) &= t(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= (x_1 + x_2)^2 \\ &= x_1^2 + x_2^2 + 2x_1x_2 \end{aligned}$$

and

$$t(\mathbf{v}_1) + t(\mathbf{v}_2) = x_1^2 + x_2^2.$$

These expressions are not equal in general, so LT1 is not satisfied.

Thus t is not a linear transformation.

(c) Since $t(\mathbf{0}) = t(0,0,0) = (0,0,0,1) \neq \mathbf{0}$, it follows that t is not a linear transformation.

Solution to Exercise C85

First we show that t satisfies LT1:

$$t(\mathbf{v}_1 + \mathbf{v}_2) = t(\mathbf{v}_1) + t(\mathbf{v}_2), \quad \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3.$$

In $\mathbb{R}^3$, let $\mathbf{v}_1 = (x_1, y_1, z_1)$ and $\mathbf{v}_2 = (x_2, y_2, z_2)$. Then

$$\begin{aligned} &t(\mathbf{v}_1 + \mathbf{v}_2) \\ &= t(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= ((x_1 + x_2)\cos\theta - (y_1 + y_2)\sin\theta, \\ &\qquad (x_1 + x_2)\sin\theta + (y_1 + y_2)\cos\theta, z_1 + z_2) \end{aligned}$$

and

$$\begin{aligned} &t(\mathbf{v}_1) + t(\mathbf{v}_2) \\ &= (x_1\cos\theta - y_1\sin\theta, x_1\sin\theta + y_1\cos\theta, z_1) \\ &\qquad + (x_2\cos\theta - y_2\sin\theta, x_2\sin\theta + y_2\cos\theta, z_2) \\ &= ((x_1 + x_2)\cos\theta - (y_1 + y_2)\sin\theta, \\ &\qquad (x_1 + x_2)\sin\theta + (y_1 + y_2)\cos\theta, z_1 + z_2). \end{aligned}$$

These expressions are equal, so LT1 is satisfied.

Next we show that t satisfies LT2:

$$t(\alpha\mathbf{v}) = \alpha\, t(\mathbf{v}), \quad \text{for all } \mathbf{v} \in \mathbb{R}^3,\ \alpha \in \mathbb{R}.$$

Let $\mathbf{v} = (x, y, z)$ be a vector in $\mathbb{R}^3$ and let $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} t(\alpha\mathbf{v}) &= t(\alpha x, \alpha y, \alpha z) \\ &= (\alpha x\cos\theta - \alpha y\sin\theta, \alpha x\sin\theta + \alpha y\cos\theta, \alpha z) \end{aligned}$$

and

$$\begin{aligned} \alpha\, t(\mathbf{v}) &= \alpha(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z) \\ &= (\alpha x\cos\theta - \alpha y\sin\theta, \alpha x\sin\theta + \alpha y\cos\theta, \alpha z). \end{aligned}$$

These expressions are equal, so LT2 is satisfied.

Since LT1 and LT2 are satisfied, t is a linear transformation.

Solution to Exercise C86

We use Strategy C14.

Since the zero element of P_3 is $p(x) = 0$, we have $p(2) = 0$ and thus $t(\mathbf{0}) = \mathbf{0}$; so t may be a linear transformation.

Next we check whether t satisfies LT1:

$$t(p(x) + q(x)) = t(p(x)) + t(q(x)), \quad \text{for all } p(x), q(x) \in P_3.$$

Let $p(x), q(x) \in P_3$. Then

$$t(p(x) + q(x)) = p(x) + q(x) + p(2) + q(2)$$

and

$$\begin{aligned} t(p(x)) + t(q(x)) &= p(x) + p(2) + q(x) + q(2) \\ &= p(x) + q(x) + p(2) + q(2). \end{aligned}$$

These expressions are equal, so LT1 is satisfied.

Finally, we check whether t satisfies LT2:

$$t(\alpha p(x)) = \alpha\, t(p(x)), \quad \text{for all } p(x) \in P_3,\ \alpha \in \mathbb{R}.$$

Let $p(x) \in P_3$ and $\alpha \in \mathbb{R}$. Then

$$t(\alpha p(x)) = \alpha p(x) + \alpha p(2)$$

and

$$\alpha\, t(p(x)) = \alpha(p(x) + p(2)) = \alpha p(x) + \alpha p(2).$$

These expressions are equal, so LT2 is satisfied.

Since both LT1 and LT2 are satisfied, t is a linear transformation.

Solution to Exercise C87

First we show that i_V satisfies LT1:

$$i_V(\mathbf{v}_1 + \mathbf{v}_2) = i_V(\mathbf{v}_1) + i_V(\mathbf{v}_2), \quad \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in V.$$

Let $\mathbf{v}_1, \mathbf{v}_2 \in V$. Then

$$i_V(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{v}_1 + \mathbf{v}_2$$

and

$$i_V(\mathbf{v}_1) + i_V(\mathbf{v}_2) = \mathbf{v}_1 + \mathbf{v}_2.$$

These expressions are equal, so LT1 is satisfied.

Next we show that i_V satisfies LT2:

$$i_V(\alpha\mathbf{v}) = \alpha\, i_V(\mathbf{v}), \quad \text{for all } \mathbf{v} \in V,\ \alpha \in \mathbb{R}.$$

Let $\mathbf{v} \in V$ and $\alpha \in \mathbb{R}$. Then

$$i_V(\alpha\mathbf{v}) = \alpha\mathbf{v}$$

and

$$\alpha\, i_V(\mathbf{v}) = \alpha\mathbf{v}.$$

These expressions are equal, so LT2 is satisfied.

Since LT1 and LT2 are satisfied, t is a linear transformation.

Solution to Exercise C88

We can write any vector (x, y) in $\mathbb{R}^2$ in the form

$$(x, y) = x(1, 0) + y(0, 1).$$

It follows from Theorem C39 that

$$\begin{aligned} q_\phi(x, y) &= q_\phi(x(1, 0) + y(0, 1)) \\ &= xq_\phi(1, 0) + yq_\phi(0, 1) \\ &= x(\cos 2\phi, \sin 2\phi) + y(\sin 2\phi, -\cos 2\phi) \\ &= (x\cos 2\phi + y\sin 2\phi, x\sin 2\phi - y\cos 2\phi). \end{aligned}$$

Solution to Exercise C89

(a) Here $E = \{(3, 1), (2, 1)\}$. Therefore,

$$\mathbf{v} = (3, 1) = 1(3, 1) + 0(2, 1),$$

so

$$\mathbf{v}_E = (1, 0)_E.$$

(b) Here $E = \{(1, 2), (2, 1)\}$. We must find $a, b \in \mathbb{R}$ such that

$$(3, 1) = (a, b)_E.$$

Since

$$(a, b)_E = a(1, 2) + b(2, 1) = (a + 2b, 2a + b),$$

equating corresponding coordinates gives the system

$$\begin{aligned} a + 2b &= 3 \\ 2a + b &= 1. \end{aligned}$$

Solving, we have $a = -\frac{1}{3}$ and $b = \frac{5}{3}$, so

$$\mathbf{v}_E = \left(-\tfrac{1}{3}, \tfrac{5}{3}\right)_E.$$

Solution to Exercise C90

(a) Here $E = \{1, x\}$. Therefore,

$$p(x) = 2 + 3x = 2 \times (1) + 3 \times (x),$$

so the E-coordinate representation of $p(x)$ is

$$(2, 3)_E.$$

(b) Here $E = \{1, 4 + 6x\}$. Therefore,

$$p(x) = 2 + 3x = 0 \times (1) + \tfrac{1}{2} \times (4 + 6x),$$

so the E-coordinate representation of $p(x)$ is

$$\left(0, \tfrac{1}{2}\right)_E.$$

(c) Here $E = \{2x, 1 + 4x\}$. We must find $a, b \in \mathbb{R}$ such that

$$p(x) = 2 + 3x = (a, b)_E.$$

Since

$$\begin{aligned} (a, b)_E &= a \times 2x + b \times (1 + 4x) \\ &= b + (2a + 4b)x, \end{aligned}$$

equating corresponding coefficients gives the system

$$\begin{aligned} b &= 2 \\ 2a + 4b &= 3. \end{aligned}$$

Solving, we have $a = -\frac{5}{2}$ and $b = 2$. Thus the E-coordinate representation of $p(x)$ is

$$\left(-\tfrac{5}{2}, 2\right)_E.$$

Solution to Exercise C91

(a) We have

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix},$$

so $t(1, 0) = (3, 0)$.

Similarly,

$$\begin{pmatrix}0\\1\end{pmatrix} \longmapsto \begin{pmatrix}3&0\\0&2\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix},$$

so $t(0,1) = (0,2)$.

Thus the coordinates of $t(1,0)$ form the first column of the matrix of t, and the coordinates of $t(0,1)$ form the second column of the matrix of t.

(b) We have

$$\begin{pmatrix}1\\0\end{pmatrix} \longmapsto \begin{pmatrix}\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix},$$

so $t(1,0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(1,1)$.

Similarly,

$$\begin{pmatrix}0\\1\end{pmatrix} \longmapsto \begin{pmatrix}\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix},$$

so $t(0,1) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(-1,1)$.

As in part (a), the coordinates of $t(1,0)$ form the first column of the matrix of t, and the coordinates of $t(0,1)$ form the second column of the matrix of t.

Solution to Exercise C92

We use Strategy C15.

(a) We find the images of the vectors in the domain basis $E = \{(1,0),(0,1)\}$:

$$t(1,0) = (1,0), \quad t(0,1) = (3,1).$$

We find the F-coordinates of each of these image vectors, where $F = \{(1,0),(0,1)\}$:

$$t(1,0) = (1,0)_F, \quad t(0,1) = (3,1)_F.$$

Hence the matrix of t with respect to the standard bases for the domain and codomain is

$$\mathbf{A} = \begin{pmatrix}1&3\\0&1\end{pmatrix}.$$

Thus the matrix representation of t with respect to these bases is

$$\begin{pmatrix}x\\y\end{pmatrix} \longmapsto \begin{pmatrix}1&3\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+3y\\y\end{pmatrix}.$$

(b) We find the images of the vectors in the domain basis $E = \{1, x, x^2\}$. The first basis vector is the constant polynomial $p_0(x) = 1$, for which $p_0(2) = 1$. The second basis vector is $p_1(x) = x$, for which $p_1(2) = 2$; and the third basis vector is $p_2(x) = x^2$, for which $p_2(2) = 4$. Thus

$$t(1) = 1+1 = 2, \quad t(x) = x+2,$$
$$t(x^2) = x^2 + 2^2 = x^2 + 4.$$

We find the F-coordinates of each of these image vectors, where $F = \{1, x, x^2\}$:

$$t(1) = (2,0,0)_F, \quad t(x) = (2,1,0)_F,$$
$$t(x^2) = (4,0,1)_F.$$

Hence the matrix of t with respect to the standard bases for the domain and codomain is

$$\mathbf{A} = \begin{pmatrix}2&2&4\\0&1&0\\0&0&1\end{pmatrix}.$$

Thus the matrix representation of t with respect to these bases is

$$\begin{pmatrix}a\\b\\c\end{pmatrix} \longmapsto \begin{pmatrix}2&2&4\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}2a+2b+4c\\b\\c\end{pmatrix}.$$

(Notice that

$$\begin{aligned}t(a+bx+cx^2) &= (a+bx+cx^2) + (a+2b+2^2c)\\ &= a+bx+cx^2+a+2b+4c\\ &= 2a+2b+4c+bx+cx^2.)\end{aligned}$$

(c) We find the images of the vectors in the domain basis $E = \{(1,0),(0,1)\}$:

$$t(1,0) = (1,0,1,0), \quad t(0,1) = (0,1,0,1).$$

We find the F-coordinates of each of these image vectors, where
$F = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$:

$$t(1,0) = (1,0,1,0)_F, \quad t(0,1) = (0,1,0,1)_F.$$

Hence the matrix of t with respect to the standard bases for the domain and codomain is

$$\mathbf{A} = \begin{pmatrix}1&0\\0&1\\1&0\\0&1\end{pmatrix}.$$

Thus the matrix representation of t with respect to these bases is

$$\begin{pmatrix}x\\y\end{pmatrix} \longmapsto \begin{pmatrix}1&0\\0&1\\1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\\x\\y\end{pmatrix}.$$

(d) We find the images of the vectors in the domain basis $E = \{(1,0,0),(0,1,0),(0,0,1)\}$:

$$t(1,0,0) = (1,0), \quad t(0,1,0) = (0,1),$$
$$t(0,0,1) = (0,0).$$

We find the F-coordinates of each of these image vectors, where $F = \{(1,0),(0,1)\}$:

$$t(1,0,0) = (1,0)_F, \quad t(0,1,0) = (0,1)_F$$
$$t(0,0,1) = (0,0)_F.$$

Hence the matrix of t with respect to the standard bases for the domain and codomain is

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus the matrix representation of t with respect to these bases is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution to Exercise C93

We use Strategy C15.

(a) We find the images of the vectors in the domain basis $E = \{(1,0,1),(1,0,0),(1,1,1)\}$:

$$t(1,0,1) = (1,0), \quad t(1,0,0) = (1,0),$$
$$t(1,1,1) = (1,1).$$

We find the F-coordinates of each of these image vectors, where $F = \{(1,0),(0,1)\}$:

$$t(1,0,1) = (1,0)_F, \quad t(1,0,0) = (1,0)_F,$$
$$t(1,1,1) = (1,1)_F.$$

Hence the matrix of t with respect to the bases E and F is

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) We find the images of the vectors in the domain basis $E = \{(1,0,0),(0,1,0),(0,0,1)\}$:

$$t(1,0,0) = (1,0), \quad t(0,1,0) = (0,1),$$
$$t(0,0,1) = (0,0).$$

We find the F-coordinates of each of these image vectors, where $F = \{(2,1),(1,1)\}$.

For the first image vector we need $a, b \in \mathbb{R}$ such that

$$(1,0) = (a,b)_F.$$

Since

$$(a,b)_F = a(2,1) + b(1,1) = (2a+b, a+b),$$

by equating coordinates we see that $a = 1$ and $b = -1$, so $(1,0) = (1,-1)_F$. Therefore

$$t(1,0,0) = (1,-1)_F.$$

For the second image vector we need $c, d \in \mathbb{R}$ such that

$$(0,1) = (c,d)_F.$$

Since

$$(c,d)_F = c(2,1) + d(1,1) = (2c+d, c+d),$$

by equating coordinates we obtain the system

$$\begin{aligned} 2c + d &= 0 \\ c + d &= 1. \end{aligned}$$

Solving, we have $c = -1$ and $d = 2$, so $(0,1) = (-1,2)_F$. Therefore

$$t(0,1,0) = (-1,2)_F.$$

Finally, for the third image vector we need $e, f \in \mathbb{R}$ such that

$$(0,0) = (e,f)_F.$$

Using the same method as before we have $e = f = 0$, so $(0,0) = (0,0)_F$. Therefore

$$t(0,0,1) = (0,0)_F.$$

Hence the matrix of t with respect to the bases E and F is

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \end{pmatrix}.$$

(c) We find the images of the vectors in the domain basis $E = \{(0,1,0),(1,1,1),(0,1,1)\}$:

$$t(0,1,0) = (0,1), \quad t(1,1,1) = (1,1),$$
$$t(0,1,1) = (0,1).$$

We find the F-coordinates of each of these image vectors, where $F = \{(1,3),(2,4)\}$.

For the first image vector we need $a, b \in \mathbb{R}$ such that

$$(0,1) = (a,b)_F.$$

Since

$$(a,b)_F = a(1,3) + b(2,4) = (a+2b, 3a+4b),$$

by equating coordinates we obtain the system

$$\begin{aligned} a+2b&=0\\ 3a+4b&=1. \end{aligned}$$

Solving, we have $a=1$ and $b=-\frac{1}{2}$, so

$$(0,1)=\left(1,-\tfrac{1}{2}\right)_F.$$

Therefore

$$t(0,1,0)=\left(1,-\tfrac{1}{2}\right)_F.$$

For the second image vector we need $c,d\in\mathbb{R}$ such that

$$(1,1)=(c,d)_F.$$

Since

$$(c,d)_F=c(1,3)+d(2,4)=(c+2d,3c+4d),$$

by equating coordinates we obtain the system

$$\begin{aligned} c+2d&=1\\ 3c+4d&=1. \end{aligned}$$

Solving, we have $c=-1$ and $d=1$, so $(1,1)=(-1,1)_F$. Therefore

$$t(1,1,1)=(-1,1)_F.$$

Since $t(0,1,1)=(0,1)=t(0,1,0)$, we have

$$t(0,1,1)=\left(1,-\tfrac{1}{2}\right)_F.$$

Hence the matrix of t with respect to the bases E and F is

$$\mathbf{A}=\begin{pmatrix} 1 & -1 & 1\\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix}.$$

Solution to Exercise C94

We use Strategy C15.

(a) We find the images of the polynomials in the domain basis $E=\{1,x,x^2\}$:

$$t(1)=0,\quad t(x)=1,\quad t(x^2)=2x.$$

We find the F-coordinates of each of these image vectors, where $F=\{2x,1+x\}$.

For the first image vector we have

$$t(1)=0=(0,0)_F.$$

For the second image vector we need $a,b\in\mathbb{R}$ such that

$$1=(a,b)_F.$$

Since

$$\begin{aligned} (a,b)_F&=a\times(2x)+b\times(1+x)\\ &=b+(2a+b)x, \end{aligned}$$

by equating coefficients we obtain the system

$$\begin{aligned} b&=1\\ 2a+b&=0. \end{aligned}$$

Solving, we have $a=-\frac{1}{2}$ and $b=1$, so $1=\left(-\frac{1}{2},1\right)_F$. Therefore

$$t(x)=\left(-\tfrac{1}{2},1\right)_F.$$

For the final image vector we have

$$t(x^2)=2x=(1,0)_F.$$

Hence the matrix of t with respect to the bases E and F is

$$\mathbf{A}=\begin{pmatrix} 0 & -\frac{1}{2} & 1\\ 0 & 1 & 0 \end{pmatrix}.$$

Thus the matrix representation of t with respect to the standard basis E and non-standard basis F is

$$\begin{pmatrix} a\\ b\\ c \end{pmatrix}_E \longmapsto \begin{pmatrix} 0 & -\frac{1}{2} & 1\\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a\\ b\\ c \end{pmatrix}_E=\begin{pmatrix} -\frac{1}{2}b+c\\ b \end{pmatrix}_F.$$

(b) We find the images of the polynomials in the domain basis $E=\{x,x^2,1\}$:

$$t(x)=1,\quad t(x^2)=2x,\quad t(1)=0.$$

We find the F-coordinates of each of these image vectors, where $F=\{2x,1+x\}$.

We know from part (a) that

$$\begin{aligned} t(x)&=\left(-\tfrac{1}{2},1\right)_F,\\ t(x^2)&=(1,0)_F,\\ t(1)&=(0,0)_F. \end{aligned}$$

Hence the matrix of t with respect to the bases E and F is

$$\mathbf{A}=\begin{pmatrix} -\frac{1}{2} & 1 & 0\\ 1 & 0 & 0 \end{pmatrix}.$$

Thus the matrix representation of t with respect to the non-standard bases E and F is

$$\begin{pmatrix} a\\ b\\ c \end{pmatrix}_E \longmapsto \begin{pmatrix} -\frac{1}{2} & 1 & 0\\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a\\ b\\ c \end{pmatrix}_E=\begin{pmatrix} -\frac{1}{2}a+b\\ a \end{pmatrix}_F.$$

Solution to Exercise C95

The functions in parts (a) and (d) are linear transformations since they are of the form

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (ax + by, cx + dy), \end{aligned}$$

for some $a, b, c, d \in \mathbb{R}$.

The functions in parts (b) and (c) are not linear transformations since they are not of this form.

Solution to Exercise C96

(a) We have

$$\begin{aligned} r(p(x, y)) &= r(3x + y, -x) \\ &= (3x + y, (3x + y) - x) \\ &= (3x + y, 2x + y). \end{aligned}$$

Thus $r \circ p$ is given by

$$\begin{aligned} r \circ p : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (3x + y, 2x + y). \end{aligned}$$

(b) We have

$$\begin{aligned} p(r(x, y)) &= p(x, x + y) \\ &= (3x + (x + y), -x) \\ &= (4x + y, -x). \end{aligned}$$

Thus $p \circ r$ is given by

$$\begin{aligned} p \circ r : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (4x + y, -x). \end{aligned}$$

Solution to Exercise C97

It follows from the Composition Rule that the matrix of $s \circ t$ with respect to the standard bases for the domain and codomain is

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 4 \\ 2 & 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 4 & 11 \\ 4 & 2 & 0 & 6 \\ 1 & 0 & 2 & 4 \end{pmatrix}.$$

Thus the matrix representation of $s \circ t$ with respect to the standard bases for the domain and codomain is

$$s \circ t : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} &\longmapsto \begin{pmatrix} 4 & 1 & 4 & 11 \\ 4 & 2 & 0 & 6 \\ 1 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \\ &= \begin{pmatrix} 4x + y + 4z + 11w \\ 4x + 2y + 6w \\ x + 2z + 4w \end{pmatrix}. \end{aligned}$$

Solution to Exercise C98

It follows from the Composition Rule that the matrix of $s \circ t$ with respect to the standard bases for the domain and codomain is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Thus the matrix representation of $s \circ t$ with respect to the standard bases for the domain and codomain is

$$s \circ t : P_3 \longrightarrow P_2$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_E \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}_E = \begin{pmatrix} b \\ 2c \end{pmatrix}_F.$$

As expected, this is the same as the matrix representation for s.

Solution to Exercise C99

Since

$$\begin{aligned} &s(t(x, y)) \\ &\quad = s(4x - y, -3x + y) \\ &\quad = ((4x - y) + (-3x + y), 3(4x - y) + 4(-3x + y)) \\ &\quad = (x, y) \end{aligned}$$

and

$$\begin{aligned} &t(s(x, y)) \\ &\quad = t(x + y, 3x + 4y) \\ &\quad = (4(x + y) - (3x + 4y), -3(x + y) + (3x + 4y)) \\ &\quad = (x, y), \end{aligned}$$

for each vector (x, y) in $\mathbb{R}^2$, s is the inverse function of t.

Solution to Exercise C100

(a) Since t is a linear transformation between two vector spaces of the same dimension, we use Strategy C16.

First we find a matrix representation of t. We have

$$t(1,0)=(2,4),\quad t(0,1)=(1,2).$$

Hence the matrix representation of t with respect to the standard bases for the domain and codomain is

$$\begin{pmatrix}x\\y\end{pmatrix}\longmapsto\begin{pmatrix}2&1\\4&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2x+y\\4x+2y\end{pmatrix}.$$

Next we evaluate the determinant of the matrix

$$\mathbf{A}=\begin{pmatrix}2&1\\4&2\end{pmatrix}.$$

We have

$$\det\mathbf{A}=\begin{vmatrix}2&1\\4&2\end{vmatrix}=4-4=0.$$

Since $\det\mathbf{A}=0$, t is not invertible.

(b) Since t is a linear transformation between two vector spaces of the same dimension, we use Strategy C16.

First we find a matrix representation of t. We have

$$t(1,0)=(1,3),\quad t(0,1)=(-1,1).$$

Hence the matrix representation of t with respect to the standard bases for the domain and codomain is

$$\begin{pmatrix}x\\y\end{pmatrix}\longmapsto\begin{pmatrix}1&-1\\3&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x-y\\3x+y\end{pmatrix}.$$

Next we evaluate the determinant of the matrix

$$\mathbf{A}=\begin{pmatrix}1&-1\\3&1\end{pmatrix}.$$

We have

$$\det\mathbf{A}=\begin{vmatrix}1&-1\\3&1\end{vmatrix}=1-(-3)=4.$$

Since $\det\mathbf{A}\neq 0$, t is invertible.

We now find the inverse function of t, $t^{-1}:\mathbb{R}^2\longrightarrow\mathbb{R}^2$. According to Strategy C16, t^{-1} has the matrix representation $\mathbf{v}\longmapsto\mathbf{A}^{-1}\mathbf{v}$, with respect to the standard bases for the domain and codomain. Since

$$\mathbf{A}^{-1}=\tfrac{1}{4}\begin{pmatrix}1&1\\-3&1\end{pmatrix}=\begin{pmatrix}\frac{1}{4}&\frac{1}{4}\\-\frac{3}{4}&\frac{1}{4}\end{pmatrix},$$

it follows that t^{-1} has the matrix representation

$$\begin{pmatrix}x\\y\end{pmatrix}\longmapsto\begin{pmatrix}\frac{1}{4}&\frac{1}{4}\\-\frac{3}{4}&\frac{1}{4}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{1}{4}x+\frac{1}{4}y\\-\frac{3}{4}x+\frac{1}{4}y\end{pmatrix}.$$

So t^{-1} is the linear transformation

$$\begin{aligned}t^{-1}:\mathbb{R}^2&\longrightarrow\mathbb{R}^2\\(x,y)&\longmapsto\left(\tfrac{1}{4}x+\tfrac{1}{4}y,-\tfrac{3}{4}x+\tfrac{1}{4}y\right).\end{aligned}$$

(c) Since t is a linear transformation between two vector spaces of the same dimension, we use Strategy C16.

First we find a matrix representation of t. We have

$$t(1,0,0)=(2,-1,0),\quad t(0,1,0)=(0,3,0),$$
$$t(0,0,1)=(0,0,1).$$

Hence the matrix representation of t with respect to the standard bases for the domain and codomain is

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\longmapsto\begin{pmatrix}2&0&0\\-1&3&0\\0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2x\\3y-x\\z\end{pmatrix}.$$

Next we evaluate the determinant of the matrix

$$\mathbf{A}=\begin{pmatrix}2&0&0\\-1&3&0\\0&0&1\end{pmatrix}.$$

We have

$$\begin{aligned}\det\mathbf{A}=\begin{vmatrix}2&0&0\\-1&3&0\\0&0&1\end{vmatrix}&=2\begin{vmatrix}3&0\\0&1\end{vmatrix}-0+0\\&=2\times 3=6.\end{aligned}$$

Since $\det\mathbf{A}\neq 0$, t is invertible.

We now find the inverse function of t, $t^{-1}:\mathbb{R}^3\longrightarrow\mathbb{R}^3$. According to Strategy C16, t^{-1} has the matrix representation $\mathbf{v}\longmapsto\mathbf{A}^{-1}\mathbf{v}$, with respect to the standard bases for the domain and codomain.

Using row-reduction from Unit C1, we find

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so t^{-1} has the matrix representation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x \\ \frac{1}{6}x + \frac{1}{3}y \\ z \end{pmatrix}.$$

So t^{-1} is the linear transformation

$$\begin{aligned} t^{-1} &: \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto \left(\tfrac{1}{2}x, \tfrac{1}{6}x + \tfrac{1}{3}y, z\right). \end{aligned}$$

(d) Since t is a linear transformation between two vector spaces of different dimensions, it follows from Corollary C46 that t is not invertible.

Solution to Exercise C101

The linear transformation

$$\begin{aligned} t : P_3 &\longrightarrow \mathbb{R}^3 \\ a + bx + cx^2 &\longmapsto (a, b, c) \end{aligned}$$

is one-to-one and onto and hence invertible. It is therefore an isomorphism.

(There are many other possibilities.)

Solution to Exercise C102

The vector spaces $\mathbb{R}^2$, $\mathbb{C}$ and P_2 are isomorphic, since they are all two-dimensional.

The vector spaces $\mathbb{R}^3$ and P_3 are isomorphic, since they are both three-dimensional.

Solution to Exercise C103

(a) The image set of this linear transformation is the x-axis. This is a subspace of the codomain.

(b) The image set of this linear transformation is the line $y = x$. This is a subspace of the codomain.

Solution to Exercise C104

We have

$$t(1, 0) = (1, 1), \quad t(0, 1) = (0, 0).$$

The image set of t is the line $y = x$; that is,

$$\operatorname{Im} t = \{(k, k) : k \in \mathbb{R}\}.$$

Thus $\operatorname{Im} t$ is spanned by $(1, 1) = t(1, 0)$.

Solution to Exercise C105

We use Strategy C17.

(a) We take the standard basis $\{(1, 0), (0, 1)\}$ for the domain $\mathbb{R}^2$.

We determine the images of these basis vectors:

$$t(1, 0) = (1, 2), \quad t(0, 1) = (0, 1).$$

The set $\{(1, 2), (0, 1)\}$ is linearly independent, so it is a basis for $\operatorname{Im} t$.

Since the basis has two elements,

$$\dim(\operatorname{Im} t) = 2.$$

(b) We take the standard basis $\{1, x, x^2\}$ for the domain P_3.

We determine the images of these basis vectors:

$$t(1) = 0, \quad t(x) = 1, \quad t(x^2) = 2x.$$

The set $\{0, 1, 2x\}$ is not linearly independent since it contains the zero vector. We discard 0 to give the set $\{1, 2x\}$.

The vectors 1 and $2x$ are linearly independent, so $\{1, 2x\}$ is a basis for $\operatorname{Im} t$.

Since the basis has two elements,

$$\dim(\operatorname{Im} t) = 2.$$

(c) We take the standard basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ for the domain $\mathbb{R}^3$.

We determine the images of these basis vectors:

$$\begin{aligned} &t(1, 0, 0) = (1, 1, 1), \quad t(0, 1, 0) = (2, 0, 1), \\ &t(0, 0, 1) = (3, 1, 2). \end{aligned}$$

The set $\{(1, 1, 1), (2, 0, 1), (3, 1, 2)\}$ is not linearly independent. In fact,

$$(3, 1, 2) = (1, 1, 1) + (2, 0, 1),$$

so we discard $(3, 1, 2)$ to give the set $\{(1, 1, 1), (2, 0, 1)\}$.

The vectors $(1, 1, 1)$ and $(2, 0, 1)$ are linearly independent, so $\{(1, 1, 1), (2, 0, 1)\}$ is a basis for $\operatorname{Im} t$.

Since the basis has two elements,

$$\dim(\operatorname{Im} t) = 2.$$

(You may have chosen to discard $(1, 1, 1)$ or $(2, 0, 1)$ instead. This would still give a correct answer.)

Solution to Exercise C106

(a) We know from Exercise C105(a) that $\dim(\operatorname{Im} t) = 2$. Thus $\operatorname{Im} t$ is the whole of the two-dimensional codomain $\mathbb{R}^2$; so t is onto.

(b) We know from Exercise C105(b) that $\dim(\operatorname{Im} t) = 2$. Thus $\operatorname{Im} t$ is the whole of the two-dimensional codomain P_2; so t is onto.

(c) We know from Exercise C105(c) that $\dim(\operatorname{Im} t) = 2$. Thus $\operatorname{Im} t$ is not the whole of the three-dimensional codomain $\mathbb{R}^3$; so t is not onto.

Solution to Exercise C107

(a) For this linear transformation, $t(x, y, z) = \mathbf{0}$ if and only if $(x, 0) = (0, 0)$, that is, if and only if $x = 0$. Thus the kernel of t is the (y, z)-plane. This is a subspace of the domain $\mathbb{R}^3$.

(b) For this linear transformation, $t(x, y) = \mathbf{0}$ if and only if $(x, x) = (0, 0)$, that is, if and only if $x = 0$. Thus the kernel of t is the y-axis. This is a subspace of the domain $\mathbb{R}^2$.

Solution to Exercise C108

(a) The kernel of t is the set of vectors (x, y) in $\mathbb{R}^2$ that satisfy

$$t(x, y) = \mathbf{0},$$

that is,

$$(x, 2x + y) = (0, 0).$$

Equating coordinates, we obtain the system

$$\begin{aligned} x \quad &= 0 \\ 2x + y &= 0. \end{aligned}$$

Substituting $x = 0$ from the first equation into the second equation, we obtain $y = 0$.

So the kernel of t is

$$\operatorname{Ker} t = \{(0, 0)\}.$$

Since this contains only the zero vector,

$$\dim(\operatorname{Ker} t) = 0,$$

that is, $\operatorname{Ker} t$ is a zero-dimensional subspace of the domain $\mathbb{R}^2$.

(b) The kernel of t is the set of vectors (x, y, z) in $\mathbb{R}^3$ that satisfy

$$t(x, y, z) = \mathbf{0},$$

that is,

$$(x + 2y + 3z, x + z, x + y + 2z) = (0, 0, 0).$$

Equating coordinates, we obtain the system

$$\begin{aligned} x + 2y + 3z &= 0 \\ x \qquad + \; z &= 0 \\ x + \; y + 2z &= 0. \end{aligned}$$

To solve this system we row-reduce the augmented matrix.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right) \quad \begin{array}{r} 6 \\ 2 \\ 4 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right) \quad \begin{array}{r} 6 \\ -4 \\ -2 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to -\tfrac{1}{2}\mathbf{r}_2 \\ \\ \end{array} \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right) \quad \begin{array}{r} 6 \\ 2 \\ -2 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 + \mathbf{r}_2 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{r} 2 \\ 2 \\ 0 \end{array}$$

The augmented matrix is in row-reduced form and we have

$$\begin{aligned} x \quad + z &= 0 \\ y + z &= 0. \end{aligned}$$

Assigning the parameter k to the unknown z, we obtain

$$x = -k, \quad y = -k, \quad z = k.$$

So the kernel of t is

$$\operatorname{Ker} t = \{(-k, -k, k) : k \in \mathbb{R}\},$$

that is, $\operatorname{Ker} t$ is the line through $(0, 0, 0)$ and $(-1, -1, 1)$.

Thus

$$\dim(\operatorname{Ker} t) = 1,$$

that is, $\operatorname{Ker} t$ is a one-dimensional subspace of the domain $\mathbb{R}^3$.

Solution to Exercise C109

Let $p(x) = a + bx + cx^2$ be a polynomial in P_3. Then

$$t(p(x)) = b + 2cx.$$

The kernel of t is the set of polynomials $p(x) = a + bx + cx^2$ in P_3 that satisfy

$$t(p(x)) = \mathbf{0},$$

that is,

$$b + 2cx = 0.$$

Equating coefficients, we obtain the system

$$\begin{aligned} b \quad &= 0 \\ 2c &= 0. \end{aligned}$$

So a can take any real value, $b = 0$ and $c = 0$.

Thus the kernel of t is

$$\operatorname{Ker} t = \{p(x) : p(x) = a,\ a \in \mathbb{R}\},$$

that is, the set of constant polynomials.

A basis for this subspace (the kernel) is $\{1\}$, so it follows that

$$\dim(\operatorname{Ker} t) = 1.$$

Solution to Exercise C110

(a) The kernel of t is $\operatorname{Ker} t = \{\mathbf{0}\}$. Thus t is one-to-one.

(b) The kernel of t is $\operatorname{Ker} t \neq \{\mathbf{0}\}$. Thus t is not one-to-one.

(c) The kernel of t is $\operatorname{Ker} t \neq \{\mathbf{0}\}$. Thus t is not one-to-one.

Solution to Exercise C111

(a) For the linear transformation

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x, 2x + y), \end{aligned}$$

we found in Exercise C105(a) that $\dim(\operatorname{Im} t) = 2$, and in Exercise C108(a) that $\dim(\operatorname{Ker} t) = 0$. Thus

$$\dim(\operatorname{Im} t) + \dim(\operatorname{Ker} t) = 2 + 0 = 2,$$

which is the dimension of the domain $\mathbb{R}^2$.

(b) For the linear transformation

$$\begin{aligned} t : P_3 &\longrightarrow P_2 \\ p(x) &\longmapsto p'(x), \end{aligned}$$

we found in Exercise C105(b) that $\dim(\operatorname{Im} t) = 2$, and in Exercise C109 that $\dim(\operatorname{Ker} t) = 1$. Thus

$$\dim(\operatorname{Im} t) + \dim(\operatorname{Ker} t) = 2 + 1 = 3,$$

which is the dimension of the domain P_3.

(c) For the linear transformation

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (x + 2y + 3z, x + z, x + y + 2z), \end{aligned}$$

we found in Exercise C105(c) that $\dim(\operatorname{Im} t) = 2$, and in Exercise C108(b) that $\dim(\operatorname{Ker} t) = 1$. Thus

$$\dim(\operatorname{Im} t) + \dim(\operatorname{Ker} t) = 2 + 1 = 3,$$

which is the dimension of the domain $\mathbb{R}^3$.

Solution to Exercise C112

(a) In this case, the dimension of the codomain (which is 3) is greater than the dimension of the domain (which is 2), so t is not onto.

(b) In this case, the codomain and the domain both have dimension 2. There are two possibilities: *either* t is both one-to-one and onto, *or* t is neither one-to-one nor onto.

(c) In this case, the dimension of the codomain (which is 2) is less than the dimension of the domain (which is 3), so t is not one-to-one.

Solution to Exercise C113

The number of solutions of this system of equations is the same as the number of vectors that map to $(1, 1, 1)$ under the linear transformation

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (x + 2y + 3z, x + z, x + y + 2z). \end{aligned}$$

We know from the solution to Exercise C105(c) that $(1, 1, 1)$ is in the image set of t, and from

Exercise C108(b) that $\operatorname{Ker} t \neq \{\mathbf{0}\}$. Thus the system of equations has infinitely many solutions.

Solution to Exercise C114

(a) This is a system of two linear equations in three unknowns. Since $3 > 2$, the system has either no solutions or infinitely many solutions.

(b) This is a system of three linear equations in three unknowns. There are two possibilities:

- the system has exactly one solution for each set of values of a, b and c
- there are some values of a, b and c for which the system has no solutions; for all other values of a, b and c, the system has infinitely many solutions.

Unit C4

Eigenvectors

Introduction

By now you should be familiar with a wide variety of linear transformations from one vector space to another, and should appreciate that the matrix of a linear transformation depends on the bases chosen for the domain and codomain. In this final unit on linear algebra we concentrate on linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$, from $\mathbb{R}^3$ to $\mathbb{R}^3$ and, more generally, from $\mathbb{R}^n$ to $\mathbb{R}^n$, and address the following question.

> Is it possible to find a basis for both the domain and codomain so that the matrix of a linear transformation is a diagonal matrix?

In the preceding units of this book you have studied vectors, matrices, vector spaces and linear transformations. The method for finding a diagonal matrix of a linear transformation (if such a matrix exists) links all these topics together. To round off the linear algebra topic, we use linear transformations and diagonal matrices to classify conics and quadrics.

1 Eigenvalues and eigenvectors

In this section you will see that some lines through the origin are mapped to themselves by some linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$: the individual points on these lines are usually moved, but, for a given line, all the points are scaled by a constant factor. You will see that this idea of fixed lines also applies to linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ and, more generally, from $\mathbb{R}^n$ to $\mathbb{R}^n$. You will learn how determinants can be used for finding these fixed lines of linear transformations.

1.1 What is an eigenvector?

In Subsection 1.1 of Unit C3 *Linear transformations* you saw that a linear transformation $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ moves the points of the plane around, but fixes the origin. Furthermore, parallel lines get mapped to parallel lines. In this section we will observe that t may map some lines through the origin onto themselves. These 'unchanged' lines are rather special.

Consider the linear transformation $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$t(x, y) = (x + 4y, x - 2y).$$

We know that t maps the origin $(0, 0)$ to itself, since this is a property of all linear transformations.

We can calculate the image of the point $(1, 0)$:

$$t(1, 0) = (1 + (4 \times 0), 1 - (2 \times 0)) = (1, 1).$$

Since linear transformations map lines through the origin to lines through the origin, t maps the line joining the points $(0,0)$ and $(1,0)$ to the line joining the points $(0,0)$ and $(1,1)$, as illustrated in Figure 1; that is,

t maps the line $y = 0$ to the line $y = x$.

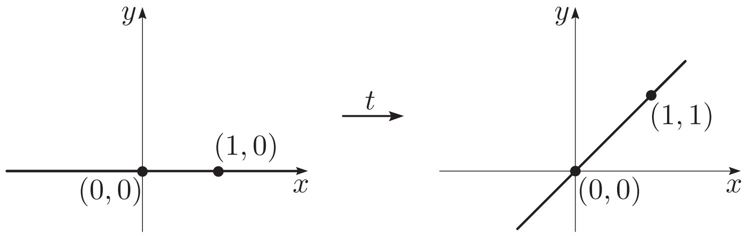


Figure 1 The image of the line $y = 0$ under the linear transformation t

Let us now calculate the image of the point $(1,-1)$:

$$t(1,-1) = (1 + 4(-1), 1 - 2(-1)) = (-3,3).$$

In this case, the linear transformation t maps the line joining the points $(0,0)$ and $(1,-1)$ to the line joining the points $(0,0)$ and $(-3,3)$, as illustrated in Figure 2; that is,

t maps the line $y = -x$ to itself.

Although t moves individual points on the line (except $(0,0)$) to other points, the line *as a whole* is unchanged.

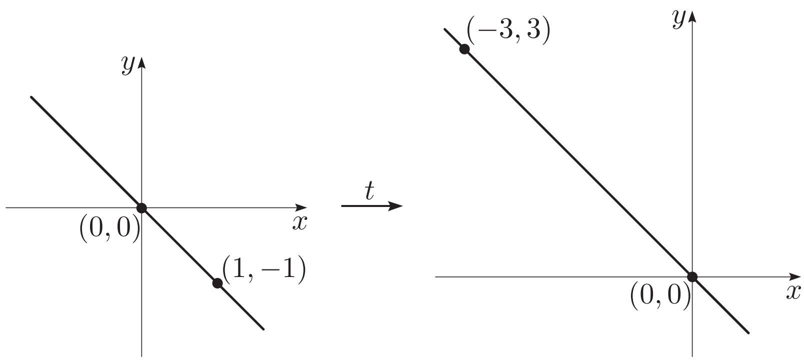


Figure 2 The image of the line $y = -x$ under the linear transformation t

The image of the point $(1,-1)$ under t is the point $(-3,3) = -3(1,-1)$. The vector $(1,-1)$ is scaled (stretched) by a factor of -3; that is, the resulting vector is three times the original magnitude and pointing in the opposite direction. In the next exercise you will investigate how other vectors lying along the line $y = -x$ are moved by t.

Exercise C115

For the above linear transformation t, calculate the images of the vectors $(2,-2)$ and $(-7,7)$. What do you notice?

We have seen that the linear transformation t scales some vectors lying along the line $y = -x$ by the factor -3. In fact this is true of any vector lying along this line, as we now show.

Let k be any real number, so that $(k, -k) = k(1, -1)$ is a vector lying along the line $y = -x$. Then

$$t(k, -k) = (k - 4k, k + 2k) = (-3k, 3k) = -3(k, -k),$$

which shows that t has the same scaling effect on each vector $(k, -k)$ lying along the line $y = -x$.

Does the linear transformation t map other lines through the origin to themselves?

Exercise C116

(a) For the above linear transformation t, calculate $t(0, 1)$, $t(1, 2)$ and $t(4, 1)$.

(b) Use one of the solutions to part (a) to write down another line in $\mathbb{R}^2$ that is mapped to itself by the linear transformation t.

(c) Find $t(4k, k)$.

We have seen that the linear transformation t maps each of the lines $y = -x$ and $x = 4y$ to itself. In both cases, each vector along the line is moved to a scalar multiple of itself: each vector lying along the line $y = -x$ is mapped to -3 times itself and each vector lying along the line $x = 4y$ is mapped to 2 times itself. We call the non-zero vectors lying along the line $y = -x$ *eigenvectors* of t with corresponding *eigenvalue* -3; for example, $(1, -1)$ and $(-7, 7)$ are eigenvectors of t with corresponding eigenvalue -3. Similarly, we call the non-zero vectors lying along the line $x = 4y$ eigenvectors of t with corresponding eigenvalue 2; for example, $(4, 1)$ and $(-8, -2)$ are eigenvectors of t with corresponding eigenvalue 2.

More generally, we make the following definitions; here and throughout this unit we use V to denote a finite-dimensional vector space.

Definitions

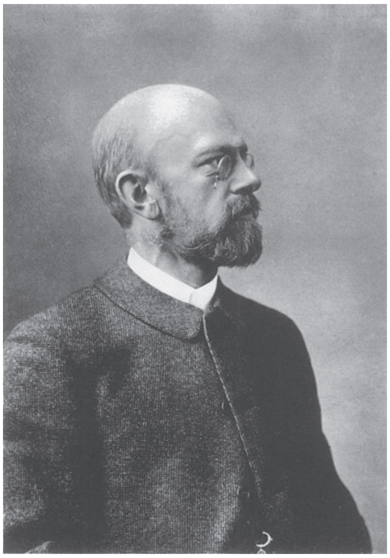
Let $t : V \longrightarrow V$ be a linear transformation. An **eigenvector** of t is a non-zero vector $\mathbf{v}$ that is mapped by t to a scalar multiple of itself; this scalar is the corresponding **eigenvalue**.

In symbols, a non-zero vector $\mathbf{v}$ is an eigenvector of a linear transformation t if

$$t(\mathbf{v}) = \lambda\mathbf{v}, \quad \text{for some } \lambda \in \mathbb{R};$$

λ is the corresponding eigenvalue.

We exclude the case $\mathbf{v} = \mathbf{0}$, since $t(\mathbf{0}) = \mathbf{0}$ for *every* linear transformation t. It is, however, possible for λ to be 0: when $\lambda = 0$, the linear transformation maps every vector corresponding to this eigenvalue to the origin – you will see an instance of this in Exercise C120.



David Hilbert

Werner Heisenberg

Eigen is a German word meaning *own*, *characteristic* or *special*. Another name for eigenvalue is *characteristic value*.

The *eigen* terms are associated with the German mathematician David Hilbert (1862–1943) who first used the terms *Eigenfunktion* (eigenfunction) and *Eigenwert* (Eigenvalue) in a series of papers on integral equations (1904–1910). It is possible that Hilbert was following the German physicist Hermann von Helmholtz (1821–1894) who used the term *Eigentöne* in acoustics in the nineteenth century.

In the 1920s the use of the eigen terminology was promoted through the development of the matrix mechanics formulation of quantum theory by the German physicist Werner Heisenberg (1901–1976) who wrote the new theory in the language of Hilbert and his followers.

In the example above we found two lines that are mapped to themselves by t, by considering the images of various points. This is a rather hit-and-miss way of finding eigenvalues and eigenvectors. Before developing a general method for finding them, we see that it is sometimes possible to do so by considering the geometry of the transformation.

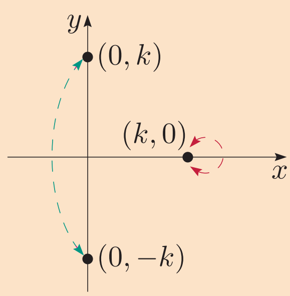
Worked Exercise C62

Let $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation that maps each point to its reflection in the x-axis. By considering the geometric features of t, determine as many eigenvectors of t as you can and write down the corresponding eigenvalue in each case.

Solution

Reflection in the x-axis maps each point (x, y) to the point $(x, -y)$.

A sketch can help.



It therefore maps each point on the x-axis to itself, since

$$t(k, 0) = (k, 0) = 1(k, 0),$$

so the vectors $(k, 0)$, with $k \neq 0$, are eigenvectors of t with corresponding eigenvalue 1.

Note that $k \neq 0$, since we exclude the zero vector.

Similarly, each point on the y-axis is mapped to minus itself, since

$$t(0, k) = (0, -k) = -1(0, k),$$

so the vectors $(0, k)$, with $k \neq 0$, are eigenvectors of t with corresponding eigenvalue -1.

It is clear geometrically that every other line through the origin is changed by this transformation: we can find no other vector that maps to a multiple of itself. The only eigenvectors of t are therefore of the forms $(k, 0)$ with eigenvalue 1 and $(0, k)$ with eigenvalue -1, where $k \neq 0$.

Exercise C117

By considering the geometric features of each of the following linear transformations of the plane, determine as many eigenvectors as you can and write down the corresponding eigenvalue in each case:

(a) reflection in the line $y = x$

(b) 2-dilation

(c) anticlockwise rotation through $\pi/2$ about the origin

(d) anticlockwise rotation through π about the origin.

In Exercise C117 it is possible to spot the eigenvectors geometrically. We now illustrate a general method to determine the eigenvalues and eigenvectors for any given transformation.

Consider again the linear transformation $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$t(x, y) = (x + 4y, x - 2y).$$

We wish to find those vectors (x, y) that are mapped to scalar multiples of themselves; that is,

$$t(x, y) = \lambda(x, y) = (\lambda x, \lambda y).$$

We equate the expressions for $t(x, y)$ and obtain

$$(x + 4y, x - 2y) = (\lambda x, \lambda y).$$

Equating the first and second coordinates of these vectors, we obtain the system of linear equations

$$\begin{aligned} x + 4y &= \lambda x \\ x - 2y &= \lambda y. \end{aligned}$$

This is a system of two equations in the three unknowns x, y and λ. One way of solving this system is to move the terms on the right to the left-hand side. Thus we obtain the system

$$\begin{aligned} (1-\lambda)x + \qquad 4y &= 0 \\ x + (-2-\lambda)y &= 0. \end{aligned} \qquad (1)$$

Equations (1) are called the *eigenvector equations*. We use them to find the possible values of λ, and then to find all the eigenvectors that correspond to these values. They are *homogeneous* equations in x and y since the constant terms are all zero.

Systems of homogeneous linear equations always have the trivial solution, in this case $x = 0$, $y = 0$, but this corresponds to the zero vector, which is excluded. Thus we seek *non-zero* solutions to the pair of homogeneous equations (1). Since we have two equations in three unknowns, such a system is bound to be dependant; that is, the homogeneous system has insufficient constraints on the unknowns to determine them uniquely.

From Theorem C19, Summary Theorem, in Unit C1 *Linear equations and matrices* we know that a homogeneous system has only the trivial solution if and only if the determinant of the coefficient matrix is non-zero. The contrapositive of this tells us that non-zero solutions exist if and only if the determinant of the coefficient matrix is 0; that is, if and only if

$$\begin{vmatrix} 1-\lambda & 4 \\ 1 & -2-\lambda \end{vmatrix} = 0.$$

We expand the determinant and obtain

$$(1-\lambda)(-2-\lambda) - 4 = 0,$$

which simplifies (after some algebra) to

$$\lambda^2 + \lambda - 6 = 0.$$

This equation is called the *characteristic equation* of t, and its solutions are the eigenvalues we seek. Notice that the characteristic equation, whether or not it is written in terms of a determinant, is a polynomial equation in λ whose degree is the dimension of the domain of t – in this case 2. Here, we have

$$\lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3) = 0,$$

so the eigenvalues are $\lambda = 2$ and $\lambda = -3$.

To find the corresponding eigenvectors, we consider each eigenvalue λ in turn.

$\boxed{\lambda = 2}$ Putting $\lambda = 2$ into the eigenvector equations (1), we obtain

$$\begin{aligned} -x + 4y &= 0 \\ x - 4y &= 0. \end{aligned}$$

One equation is -1 times the other, so the equations are equivalent to the single equation

$$x = 4y.$$

Thus the eigenvectors corresponding to $\lambda = 2$ are the non-zero vectors (x, y) for which $x = 4y$; that is, the vectors of the form

$$(4k, k), \quad \text{where } k \neq 0.$$

Since we are working in a real vector space, in this case $\mathbb{R}^2$, when we are talking about eigenvectors, k represents a real number.

$\boxed{\lambda = -3}$ Putting $\lambda = -3$ into the eigenvector equations (1), we obtain

$$\begin{aligned} 4x + 4y &= 0 \\ x + y &= 0. \end{aligned}$$

These equations are equivalent to the single equation

$$y = -x.$$

Thus the eigenvectors corresponding to $\lambda = -3$ are the non-zero vectors (x, y) for which $y = -x$; that is, the vectors of the form

$$(k, -k), \quad \text{where } k \neq 0.$$

Thus the eigenvectors of t are the non-zero vectors of the following forms:

$$\begin{aligned} &(4k, k), \text{ corresponding to } \lambda = 2, \\ &(k, -k), \text{ corresponding to } \lambda = -3. \end{aligned}$$

This method produces *all* the eigenvalues and eigenvectors of the linear transformation. On the other hand, trying to show that these are the only ones by calculating the images of various points, as we started to do at the beginning of the section, would take forever!

Exercise C118

Let $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation given by

$$t(x, y) = (-5x + 3y, 6x - 2y).$$

(a) Find the eigenvector equations of t.

(b) Find the characteristic equation of t, and solve it to find the eigenvalues of t.

(c) Solve the eigenvector equations, for each eigenvalue in turn, to find the eigenvectors of t.

1.2 Finding eigenvalues and eigenvectors

You have just seen how to find the eigenvalues and eigenvectors of a given linear transformation $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. This method, as it stands, is rather tedious to use to find eigenvalues and eigenvectors of linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$, or $\mathbb{R}^4$ to $\mathbb{R}^4$, and so on. However, by introducing matrices, we can simplify the method.

We now work through the same example as in the previous subsection, but this time we use matrices.

Theorem C40 of Unit C3 tells us that there is a unique matrix for t with respect to the standard (ordered) basis in both the domain and codomain, and we use Strategy C15 from that unit to find this matrix. Recall that this strategy tells us essentially to 'read off' the matrix of a linear transformation when we are using the standard bases. We have $t(1,0) = (1,1)$ and $t(0,1) = (4,-2)$, so these vectors are the columns of the matrix of the linear transformation, since we are using the standard bases.

Therefore, with respect to the standard basis for $\mathbb{R}^2$, the linear transformation t given by $t(x,y) = (x+4y, x-2y)$ has the matrix representation

$$t : \mathbf{v} \longmapsto \mathbf{A}\mathbf{v}, \quad \text{where } \mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}.$$

If $\mathbf{v}$ is an eigenvector of t with corresponding eigenvalue λ, then

$$t(\mathbf{v}) = \lambda\mathbf{v};$$

in matrix form, this becomes

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v};$$

that is,

$$\begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}.$$

So

$$\begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \lambda\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (2)$$

Using the 2×2 identity matrix $\mathbf{I}$, we can write

$$\lambda\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

so equation (2) can be written as

$$\left[\begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

that is,

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}.$$

We simplify this matrix equation and obtain

$$\begin{pmatrix} 1-\lambda & 4 \\ 1 & -2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives rise to the eigenvector equations

$$\begin{aligned} (1-\lambda)x + 4y &= 0 \\ x + (-2-\lambda)y &= 0, \end{aligned}$$

as before, which we labelled equations (1). The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 4 \\ 1 & -2-\lambda \end{vmatrix} = 0,$$

that is,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

We can therefore find the characteristic equation directly from the matrix of the linear transformation (with respect to the standard basis for both the domain and codomain) by subtracting λ from each diagonal entry and then equating the determinant to zero.

Once we have found the eigenvalues, we use the same method as before to find the eigenvectors; that is, we substitute each eigenvalue in turn into the eigenvector equations and solve them.

In view of this connection with matrices, we adopt the following definitions.

Definitions

A non-zero vector $\mathbf{v}$ is an **eigenvector** of a square matrix $\mathbf{A}$ if

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}, \quad \text{for some } \lambda \in \mathbb{R};$$

λ is the corresponding **eigenvalue**.

The **characteristic equation** of a square matrix $\mathbf{A}$ is the equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

In this way we can refer to eigenvectors, eigenvalues and the characteristic equation of a matrix even when a linear transformation is not explicitly involved.

The matrix $\mathbf{A} - \lambda\mathbf{I}$ is obtained by subtracting λ from each entry on the diagonal of $\mathbf{A}$.

Eigenvalues and eigenvectors of matrices occur naturally in many applications – for example, in the study of vibrating mechanical systems. In such examples, the characteristic equation may have solutions that are not real numbers, and these *complex eigenvalues* have significance in these applications. In this unit we are primarily interested in linear transformations of the plane and of three-dimensional space, so complex eigenvalues play no role here: we are concerned only with *real* eigenvalues and eigenvectors.

Other areas of application include music, bridge design, oil exploration, image compression, and analysis of financial data. A particular example is the use of eigenvectors in the PageRank algorithm. This algorithm was invented by Larry Page and Sergey Brin, the founders of Google, in 1996 for use by the Google search engine to rank the importance of web pages. According to Google, PageRank works by counting the number and quality of links to a page to determine a rough estimate of how important the website is. The underlying assumption is that more important websites are likely to receive more links from other websites. The algorithm assigns a PageRank, or score, to each web page based on its linking web pages, with the links from different web pages being weighted according to particular criteria. The Google matrix represents the links between the web pages. A fundamental part of the algorithm is an iterative method that computes the dominant eigenvalue, that is, the eigenvalue of largest magnitude, and the corresponding eigenvector of the Google matrix to rank the web pages.

Larry Page and Sergey Brin

If a characteristic equation has no real solutions, then we say that there are no eigenvalues. For example, in Exercise C117(c), you considered the linear transformation representing an anticlockwise rotation through $\pi/2$ about the origin. The matrix of this linear transformation is

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

By the above definition, the characteristic equation of this linear transformation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = 0.$$

We expand the determinant and obtain

$$\lambda^2 + 1 = 0.$$

This equation has no real solutions: the linear transformation has no eigenvalues and hence no eigenvectors. This agrees with the geometric interpretation: no line through the origin is mapped to itself by this rotation.

We summarise this matrix method for finding eigenvalues and eigenvectors in the following strategy.

Strategy C18

To determine the eigenvalues and eigenvectors of a square matrix $\mathbf{A}$, do the following.

1. Find the eigenvalues:
   - write down the characteristic equation
     $$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$
   - expand this determinant to obtain a polynomial equation in λ
   - solve this equation to find the eigenvalues.
2. Find the eigenvectors:
   - write down the eigenvector equations
     $$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$
   - for each eigenvalue λ, solve this system of linear equations to find the corresponding eigenvectors.

We illustrate Strategy C18 with the following worked exercise and exercise.

Worked Exercise C63

Let $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation given by

$$t(x, y) = (5x + 2y, 2x + 5y).$$

Write down the matrix of t with respect to the standard basis for $\mathbb{R}^2$, and find the eigenvalues and eigenvectors of t.

Solution

Since we are using the standard bases, we can simply 'read off' the matrix: the columns are the images of $(1, 0)$ and $(0, 1)$ under t.

The matrix of t with respect to the standard basis for $\mathbb{R}^2$ is

$$\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}.$$

We use Strategy C18 to find the eigenvalues and eigenvectors of $\mathbf{A}$, which are the same as those of t.

First we find the eigenvalues of $\mathbf{A}$.

The characteristic equation of $\mathbf{A}$ is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$; that is,

$$\begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = 0.$$

We expand the determinant and obtain

$$(5 - \lambda)^2 - 4 = 0,$$

which simplifies to

$$\lambda^2 - 10\lambda + 21 = (\lambda - 7)(\lambda - 3) = 0.$$

The eigenvalues of $\mathbf{A}$ are therefore $\lambda = 7$ and $\lambda = 3$.

Next we find the eigenvectors of $\mathbf{A}$.

The eigenvector equations are $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$; that is,

$$\begin{pmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$$

which we write as a system of linear equations.

The eigenvector equations are

$$\begin{aligned} (5-\lambda)x + 2y &= 0 \\ 2x + (5-\lambda)y &= 0. \end{aligned}$$

$\boxed{\lambda = 7}$ The eigenvector equations become

$$\begin{aligned} -2x + 2y &= 0 \\ 2x - 2y &= 0. \end{aligned}$$

These equations are equivalent to the single equation

$$y = x.$$

Thus the eigenvectors corresponding to $\lambda = 7$ are the non-zero vectors for which $y = x$; that is, the vectors of the form

$$(k, k), \quad \text{where } k \neq 0.$$

$\boxed{\lambda = 3}$ The eigenvector equations become

$$\begin{aligned} 2x + 2y &= 0, \\ 2x + 2y &= 0. \end{aligned}$$

These equations are equivalent to the single equation

$$y = -x.$$

Thus the eigenvectors corresponding to $\lambda = 3$ are the non-zero vectors for which $y = -x$; that is, the vectors of the form

$$(k, -k), \quad \text{where } k \neq 0.$$

Thus the eigenvectors of the linear transformation t are the non-zero vectors of the following forms:

$$\begin{aligned} &(k, k), \text{ corresponding to } \lambda = 7, \\ &(k, -k), \text{ corresponding to } \lambda = 3. \end{aligned}$$

Exercise C119

For each of the following linear transformations $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, write down the matrix of t with respect to the standard basis for $\mathbb{R}^2$, and find the eigenvalues and eigenvectors of t.

(a) $t(x, y) = (x + 3y, 2x - 4y)$ (b) $t(x, y) = (x - 2y, -2x - 2y)$

So far we have concentrated on linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ and on 2×2 matrices. We now use Strategy C18 to find the eigenvalues and eigenvectors of a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ using a 3×3 matrix. Notice that here the characteristic equation is again a polynomial equation in λ whose degree is the dimension of the domain of t – in this case 3.

Worked Exercise C64

Let $t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation given by

$$t(x, y, z) = (2x + z, -x + 2y + 3z, x + 2z).$$

Write down the matrix of t with respect to the standard basis for $\mathbb{R}^3$, and find the eigenvalues and eigenvectors of t.

Solution

Since we are using the standard basis, we can again simply 'read off' the matrix: the columns are the images of $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ under t.

The matrix of t with respect to the standard basis for $\mathbb{R}^3$ is

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix}.$$

We use Strategy C18 to find the eigenvalues and eigenvectors of $\mathbf{A}$, which are the same as those of t.

First we find the eigenvalues of $\mathbf{A}$.

Here we need the 3×3 identity matrix $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and so subtract λ from the three diagonal entries of $\mathbf{A}$.

The characteristic equation is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$; that is,

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ -1 & 2-\lambda & 3 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0.$$

We expand the determinant and obtain

$$(2-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} - 0 + \begin{vmatrix} -1 & 2-\lambda \\ 1 & 0 \end{vmatrix} = 0.$$

Simplifying this expression, we obtain

$$(2-\lambda)((2-\lambda)^2 - 0) + (0 - (2-\lambda)) = 0.$$

When there is a common factor, it is best to keep this separate: the problem then reduces to factorising the remaining quadratic polynomial.

Taking out the common factor gives

$$(2-\lambda)((2-\lambda)^2 - 1) = 0,$$

which simplifies to

$$(2-\lambda)(\lambda^2 - 4\lambda + 3) = 0.$$

We can factorise this characteristic equation as

$$(2-\lambda)(\lambda-3)(\lambda-1) = 0.$$

The eigenvalues of $\mathbf{A}$ are therefore $\lambda = 3$, $\lambda = 2$ and $\lambda = 1$.

Next we find the eigenvectors of $\mathbf{A}$.

The eigenvector equations are

$$\begin{aligned} (2-\lambda)x \qquad\qquad\quad &+ \qquad\quad z = 0 \\ -x + (2-\lambda)y &+ \qquad 3z = 0 \\ x \qquad\qquad\quad &+ (2-\lambda)z = 0. \end{aligned}$$

$\boxed{\lambda = 3}$ The eigenvector equations become

$$\begin{aligned} -x \qquad + \ \ z &= 0 \\ -x - y + 3z &= 0 \\ x \qquad - \ \ z &= 0. \end{aligned}$$

It may sometimes be necessary to use the method of Gauss–Jordan elimination from Unit C1, but here the solutions can be found directly.

The first and third equations imply that

$$z = x.$$

Substituting this into the second equation yields the equation

$$2x - y = 0.$$

Thus the eigenvectors corresponding to $\lambda = 3$ are the non-zero vectors (x, y, z) satisfying $z = x$ and $y = 2x$; that is, the vectors of the form

$$(k, 2k, k), \quad \text{where } k \neq 0.$$

$\boxed{\lambda = 2}$ The eigenvector equations become

$$\begin{aligned} z &= 0 \\ -x + 3z &= 0 \\ x \quad &= 0. \end{aligned}$$

These equations have the solution

$$z = 0 \quad \text{and} \quad x = 0.$$

However, there are no constraints on the unknown y. Thus the eigenvectors corresponding to $\lambda = 2$ are the non-zero vectors (x, y, z) satisfying $x = 0$ and $z = 0$; that is, the vectors of the form

$$(0, k, 0), \quad \text{where } k \neq 0.$$

$\boxed{\lambda = 1}$ The eigenvector equations become

$$\begin{aligned} x \quad + z &= 0 \\ -x + y + 3z &= 0 \\ x \quad + z &= 0. \end{aligned}$$

The first and third equations imply that

$$z = -x.$$

Substituting this into the second equation yields the equation

$$-4x + y = 0.$$

Thus the eigenvectors corresponding to $\lambda = 1$ are the non-zero vectors (x, y, z) satisfying $z = -x$ and $y = 4x$; that is, the vectors of the form

$$(k, 4k, -k), \quad \text{where } k \neq 0.$$

Thus the eigenvectors of the linear transformation t are the non-zero vectors of the following forms:

$$\begin{aligned} &(k, 2k, k), \text{ corresponding to } \lambda = 3, \\ &(0, k, 0), \text{ corresponding to } \lambda = 2, \\ &(k, 4k, -k), \text{ corresponding to } \lambda = 1. \end{aligned}$$

Although cubic polynomials may not always be easy to factorise, you met some ways of factorising such polynomials in Subsection 1.4 of Unit A2 *Number systems*. However, we will usually deal with examples that factorise easily.

The following result, which we do not prove here, gives a useful check on the values found for the eigenvalues. You are asked to prove it yourself for 2×2 matrices in the additional exercises booklet for this unit.

Proposition C56

The sum of the eigenvalues of a square matrix $\mathbf{A}$ is equal to the sum of the diagonal entries of $\mathbf{A}$.

For example, in Worked Exercise C64 the eigenvalues are 3, 2 and 1, which sum to 6, and the diagonal entries of the matrix $\mathbf{A}$ are 2, 2 and 2, which also sum to 6.

The sum of the diagonal entries of a square matrix is sometimes referred to as the **trace** of the matrix.

Exercise C120

Let $t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation given by

$$t(x, y, z) = (4x + 2y, 2x + 3y + 2z, 2y + 2z).$$

Write down the matrix of t with respect to the standard basis for $\mathbb{R}^3$, and find the eigenvalues and eigenvectors of t.

In most of the examples we have seen so far, the eigenvalues have not been easy to recognise directly and Strategy C18 has been required to find them. This is not always the case, as the following exercise illustrates.

Exercise C121

Find the eigenvalues of each of the following matrices.

(a) $\begin{pmatrix} 1 & 2 \\ 0 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 8 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 21 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & 0 & 0 \\ 25 & -2 & 0 \\ 17 & \pi & 6 \end{pmatrix}$

Finding eigenvalues of triangular and diagonal matrices is straightforward, as Exercise C121 illustrates. The eigenvalues are the diagonal entries of the matrix and no calculation is needed to find them.

Theorem C57

The eigenvalues of a triangular matrix and of a diagonal matrix are the diagonal entries of the matrix.

Proof A lower triangular matrix has every entry above the main diagonal zero. A diagonal matrix and the transpose of an upper triangular matrix are lower triangular matrices, so we can consider just lower triangular matrices here.

Let $\mathbf{A} = (a_{ij})$ be an $n \times n$ lower triangular matrix, so $a_{ij} = 0$ for all $j > i$. The eigenvalues of $\mathbf{A}$ are the solutions to the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. Now $\mathbf{A} - \lambda\mathbf{I}$ has diagonal entries $a_{ii} - \lambda$, and every entry above the main diagonal is zero.

We expand the determinant along the top row and continue by expanding along the top row of the resulting determinants until the only determinants in the expression are of size 2×2.

The first term in the full expansion of the determinant is the only non-zero term in the expansion because of the placement of the zeros in the smaller determinants. This non-zero term is $(a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)$. Therefore the solutions to the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ are $a_{11}, a_{22}, \ldots, a_{nn}$, by the Factor Theorem (Theorem A2 in Unit A2), and the eigenvalues of $\mathbf{A}$ are precisely the diagonal entries of the matrix.

A diagonal matrix is lower triangular and $\det \mathbf{A}^T = \det \mathbf{A}$, so the eigenvalues of a triangular or diagonal matrix are the diagonal entries. ■

1.3 Eigenspaces

In Subsection 1.1 we considered the linear transformation $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$t(x, y) = (x + 4y, x - 2y),$$

and saw that each of the lines $y = -x$ and $x = 4y$ is mapped to itself.

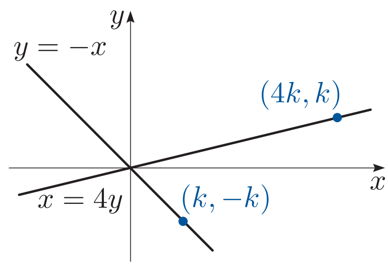


Figure 3 The lines comprising the eigenvectors of t

The line $y = -x$, shown in Figure 3, consists of the points of the form $(k, -k)$, each of which is an eigenvector of t corresponding to the eigenvalue $\lambda = -3$, except when $k = 0$, which is specifically excluded.

Similarly, the line $x = 4y$, also shown in Figure 3, consists of the points of the form $(4k, k)$, each of which is an eigenvector corresponding to the eigenvalue $\lambda = 2$, except when $k = 0$.

For each eigenvalue λ, if we look at *all* the solutions to the equation $t(\mathbf{v}) = \lambda\mathbf{v}$ (including $\mathbf{v} = \mathbf{0}$), then we obtain a line through the origin. The set of such solutions is a subspace of the domain of t.

Theorem C58

Let $t : V \longrightarrow V$ be a linear transformation. For each eigenvalue λ of t, let $S(\lambda)$ be the set of vectors satisfying $t(\mathbf{v}) = \lambda\mathbf{v}$; that is, $S(\lambda)$ is the set of eigenvectors corresponding to λ, together with the zero vector $\mathbf{0}$. Then $S(\lambda)$ is a subspace of V.

Proof Consider any eigenvalue λ of a linear transformation $t : V \longrightarrow V$.

We use Strategy C10 from Unit C2, *Vector spaces* and first check that $\mathbf{0} \in S(\lambda)$.

For any linear transformation t, we have $t(\mathbf{0}) = \mathbf{0} = \lambda\mathbf{0}$, so $\mathbf{0} \in S(\lambda)$.

Next we check that if $\mathbf{v}_1, \mathbf{v}_2 \in S(\lambda)$, then $\mathbf{v}_1 + \mathbf{v}_2 \in S(\lambda)$.

Let $\mathbf{v}_1, \mathbf{v}_2 \in S(\lambda)$. Then

$$t(\mathbf{v}_1 + \mathbf{v}_2) = t(\mathbf{v}_1) + t(\mathbf{v}_2) = \lambda\mathbf{v}_1 + \lambda\mathbf{v}_2 = \lambda(\mathbf{v}_1 + \mathbf{v}_2),$$

since t is a linear transformation.

Hence $\mathbf{v}_1 + \mathbf{v}_2 \in S(\lambda)$.

Finally, we check that if $\mathbf{v} \in S(\lambda)$ and $\alpha \in \mathbb{R}$, then $\alpha\mathbf{v} \in S(\lambda)$.

Let $\mathbf{v} \in S(\lambda)$ and $\alpha \in \mathbb{R}$. Then

$$t(\alpha\mathbf{v}) = \alpha\, t(\mathbf{v}) = \alpha\lambda\mathbf{v} = \lambda(\alpha\mathbf{v}),$$

since t is a linear transformation.

Hence $\alpha\mathbf{v} \in S(\lambda)$.

Thus $S(\lambda)$ is a subspace of V. ■

Since $S(\lambda)$ is a subspace comprising eigenvectors (and $\mathbf{0}$), we call it an *eigenspace*.

Definition

Let $t : V \longrightarrow V$ be a linear transformation and, for each eigenvalue λ of t, let $S(\lambda)$ be the set of vectors satisfying $t(\mathbf{v}) = \lambda\mathbf{v}$. Then $S(\lambda)$ is the **eigenspace** of t corresponding to the eigenvalue λ.

Worked Exercise C65

Let $t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation given by

$$t(x, y, z) = (4x + 2y, 2x + 3y + 2z, 2y + 2z).$$

Find the eigenspace $S(0)$ of t, specify a basis for it and state its dimension.

(You found the eigenvalues and eigenvectors of this linear transformation in Exercise C120.)

Solution

The non-zero vectors of the form $(k, -2k, 2k)$ are the eigenvectors of t corresponding to the eigenvalue $\lambda = 0$.

The eigenspace $S(0)$ is therefore the set of vectors

$$\{(k, -2k, 2k) : k \in \mathbb{R}\}.$$

Any vector in $S(0)$ can be written as $k(1,-2,2)$, so

$$\{(1,-2,2)\}$$

is a basis for $S(0)$. Thus $S(0)$ has dimension 1.

Geometrically, $S(0)$ is a line through the origin in the direction of the vector $(1,-2,2)$, so the only eigenvectors of t corresponding to $\lambda=0$ are on this line.

Exercise C122

Let $t:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation given by

$$t(x,y,z)=(4x+2y,2x+3y+2z,2y+2z).$$

Find the eigenspaces $S(6)$ and $S(3)$ of t. In each case, specify a basis and state the dimension of the eigenspace.

(In Exercise C120 you found that the eigenvectors of t are the non-zero vectors $(2k,2k,k)$ and $(-2k,k,2k)$, corresponding to the eigenvalues $\lambda=6$ and $\lambda=3$, respectively.)

Worked Exercise C66

Let $t:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation given by

$$t(x,y,z)=(0,y,z).$$

Find all the eigenspaces of t. In each case, specify a basis and state the dimension of the eigenspace.

Solution

The matrix of t with respect to the standard basis for $\mathbb{R}^3$ is

$$\mathbf{A}=\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

This matrix is diagonal, so the eigenvalues are the diagonal entries: $\lambda=0$, $\lambda=1$ and $\lambda=1$.

The eigenvector equations are

$$\begin{aligned}-\lambda x \qquad\qquad\qquad\qquad &= 0\\ (1-\lambda)y \qquad\qquad &= 0\\ (1-\lambda)z &= 0.\end{aligned}$$

$\boxed{\lambda = 0}$ The eigenvector equations become

$$0x = 0,\ y = 0 \text{ and } z = 0.$$

Thus the eigenvectors corresponding to the eigenvalue $\lambda = 0$ are the non-zero vectors (x, y, z) satisfying $y = 0$ and $z = 0$; that is, the vectors of the form

$$(k, 0, 0), \quad \text{where } k \neq 0.$$

The eigenspace $S(0)$ is the set of vectors

$$\{(k, 0, 0) : k \in \mathbb{R}\}.$$

Any vector in $S(0)$ can be written as $k(1, 0, 0)$, so

$$\{(1, 0, 0)\}$$

is a basis for $S(0)$. Thus $S(0)$ has dimension 1.

Geometrically, $S(0)$ is the x-axis in $\mathbb{R}^3$.

$\boxed{\lambda = 1}$ The eigenvector equations reduce to the single equation

$$-x = 0.$$

Thus the eigenvectors corresponding to the eigenvalue $\lambda = 1$ are the non-zero vectors (x, y, z) satisfying $x = 0$; that is, the vectors of the form

$$(0, k, l), \quad \text{where } k \text{ and } l \text{ are not both } 0.$$

The eigenspace $S(1)$ is the set of vectors

$$\{(0, k, l) : k, l \in \mathbb{R}\}.$$

Any vector in $S(1)$ can be written as $k(0, 1, 0) + l(0, 0, 1)$, so

$$\{(0, 1, 0), (0, 0, 1)\}$$

is a basis for $S(1)$. Thus $S(1)$ has dimension 2.

Geometrically, $S(1)$ is the plane $x = 0$ through the origin.

In Worked Exercise C66 the (simplified) characteristic equation of the linear transformation t is

$$\lambda(\lambda - 1)^2 = 0.$$

The eigenvalue $\lambda = 1$ is a 'repeated' solution of this characteristic equation; it is a *multiple root* and we say that $\lambda = 1$ has *multiplicity* 2 because the factor $(\lambda - 1)$ occurs twice.

In general, we adopt the following definition.

Definition

If the characteristic equation of a square matrix $\mathbf{A}$ can be written as

$$(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_p)^{m_p} = 0,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_p$ are distinct, then the eigenvalue λ_j of $\mathbf{A}$ has **multiplicity** m_j, for $j = 1, 2, \ldots, p$.

For a triangular or diagonal matrix, the multiplicity of an eigenvalue is the number of times it appears on the main diagonal.

Exercise C123

Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

For each eigenvalue λ, state its multiplicity, find the corresponding eigenspace $S(\lambda)$, specify a basis for $S(\lambda)$ and state its dimension.

From the examples that you have seen so far, you may be tempted to conjecture that the dimension of the eigenspace $S(\lambda)$, for a given eigenvalue λ, is equal to the multiplicity of λ. The following exercises give you the chance to investigate this conjecture.

Exercise C124

Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

For each eigenvalue λ, state its multiplicity, find the corresponding eigenspace $S(\lambda)$, specify a basis for $S(\lambda)$ and state its dimension.

Exercise C125

Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 4 & 1 \\ -1 & 1 & 4 \end{pmatrix}.$$

For each eigenvalue λ, state its multiplicity, find the corresponding eigenspace $S(\lambda)$, specify a basis for $S(\lambda)$ and state its dimension.

Hint: Look for factors in the characteristic equation and remember that $x^2 - 1 = (x-1)(x+1)$.

In Exercise C124 the eigenvalue $\lambda = 1$ has multiplicity 2, but it gives rise to an eigenspace of dimension only 1. In this case, the matrix represents a shear in the x-direction by a factor 1, as shown in Figure 4, and the only line through the origin left unchanged is the x-axis. Thus there is a single one-dimensional eigenspace, so the conjecture that the dimension of the eigenspace $S(\lambda)$ is equal to the multiplicity of λ is false.

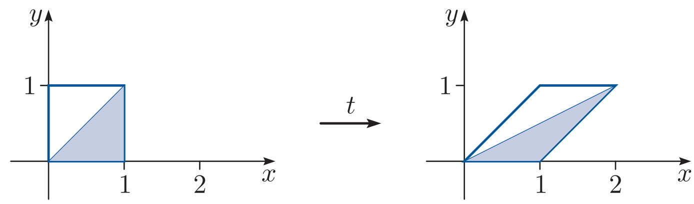


Figure 4 A shear in the x-direction by a factor 1

In Exercise C125 both eigenspaces have dimension 1 despite the eigenvalue 2 having multiplicity 2 and the eigenvalue 5 having multiplicity 1. In general, it can be shown that the dimension of an eigenspace cannot exceed the multiplicity of the corresponding eigenvalue, but we will not prove this.

2 Diagonalising matrices

In this section you will use the methods of finding eigenvalues and their corresponding eigenvectors that you met in the previous section to address the question posed in the introduction:

> Is it possible to find a basis for both the domain and codomain so that the matrix of a linear transformation is a diagonal matrix?

It is therefore important that you are confident with the material in Section 1 before starting to study this section.

2.1 Eigenvector bases

In Section 1 we introduced the notions of an eigenvalue λ and corresponding eigenvector $\mathbf{v}$ of a linear transformation $t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$; that is, a non-zero vector $\mathbf{v}$ whose image $t(\mathbf{v})$ is $\lambda\mathbf{v}$. For example, in Exercise C119(a) you saw that the linear transformation $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$t(x, y) = (x + 3y, 2x - 4y)$$

has eigenvalues $\lambda = -5$ and $\lambda = 2$ with corresponding eigenvectors the non-zero vectors of the forms $(k, -2k)$ and $(3k, k)$, respectively. We can choose any value of k $(k \neq 0)$ to specify specific eigenvectors; here, putting $k = 1$ in both gives $(1, -2)$ and $(3, 1)$. Since $(3, 1)$ is not a multiple of $(1, -2)$, these two eigenvectors are linearly independent (this is the case whatever values of k are chosen). Therefore, by Theorem C25 in Unit C2, these linearly independent eigenvectors form a basis for $\mathbb{R}^2$ – the domain and codomain of t. We say that $\{(1, -2), (3, 1)\}$ is an *eigenvector basis* of t.

Definition

Let $t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation and let E be a basis for $\mathbb{R}^n$ consisting of eigenvectors of t. The basis E is an **eigenvector basis** of t.

Exercise C126

Verify that $\{(-2,1),(1,2)\}$ is an eigenvector basis of the linear transformation $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$t(x,y) = (x-2y, -2x-2y).$$

(In Exercise C119(b) you found that the eigenvectors of t are the non-zero vectors $(-2k,k)$ and $(k,2k)$, corresponding to the eigenvalues $\lambda = 2$ and $\lambda = -3$, respectively.)

Exercise C127

The set $E = \{(0,1,-1),(-2,1,0),(1,0,-1)\}$ is a basis for $\mathbb{R}^3$. Verify that E is an eigenvector basis of the linear transformation $t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by

$$t(x,y,z) = (-x+2y+2z, 2x+2y+2z, -3x-6y-6z).$$

In Unit C3 you met Strategy C15 for finding the matrix representation of a linear transformation $t : V \longrightarrow W$ with respect to given bases E and F for the domain and codomain of t. In this subsection you will see that this matrix representation is particularly simple if $W = V$, E is an eigenvector basis of t and $F = E$.

Recall that if $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for V, and $\mathbf{v}$ is a vector in V such that $\mathbf{v} = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n$, then the numbers $v_1, \ldots, v_n$ are the E-coordinates of $\mathbf{v}$, and $\mathbf{v}_E = (v_1, \ldots, v_n)_E$ is the E-coordinate representation of $\mathbf{v}$. If E is the standard basis for V, then we usually omit the suffix E.

We begin by rewriting Strategy C15 for the particular case when $W = V$ and $F = E$ (not necessarily an eigenvector basis).

Strategy C19 (Strategy C15 with $W = V$ and $F = E$)

To find the matrix $\mathbf{A}$ of a linear transformation $t : V \longrightarrow V$ with respect to the basis $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, do the following.

1. Find $t(\mathbf{e}_1), t(\mathbf{e}_2), \ldots, t(\mathbf{e}_n)$.
2. Find the E-coordinates of each of these image vectors.
3. Construct the matrix $\mathbf{A}$ column by column using the E-coordinates of $t(\mathbf{e}_j)$ to form column j, for $j = 1, 2, \ldots, n$.

In the next worked exercise we illustrate what happens when we find the matrix of a linear transformation t with respect to an eigenvector basis of t.

Worked Exercise C67

Consider the linear transformation $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$t(x, y) = (x + 3y, 2x - 4y).$$

(a) Write down the matrix of t with respect to the standard basis for $\mathbb{R}^2$.

(b) Find the matrix of t with respect to the eigenvector basis

$$E = \{(1, -2), (3, 1)\}.$$

Solution

(a) The matrix of t with respect to the standard basis for $\mathbb{R}^2$ is

$$\begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}.$$

(b) Following Strategy C19, first we find the images of the vectors in the basis $E = \{(1, -2), (3, 1)\}$:

$$t(1, -2) = (-5, 10) \quad \text{and} \quad t(3, 1) = (6, 2).$$

We now write these image vectors in terms of their coordinates with respect to the eigenvector basis; that is, we express each of these vectors as a linear combination of the basis vectors $E = \{(1, -2), (3, 1)\}$. The resulting calculations are remarkably straightforward!

Next we find the E-coordinates of each of these image vectors:

$$\begin{aligned} (-5, 10) &= -5(1, -2) + 0(3, 1) \\ &= (-5, 0)_E, \\ (6, 2) &= 0(1, -2) + 2(3, 1) \\ &= (0, 2)_E. \end{aligned}$$

Therefore $t(1, -2) = (-5, 0)_E$ and $t(3, 1) = (0, 2)_E$. So the matrix of t with respect to the eigenvector basis E is

$$\begin{pmatrix} -5 & 0 \\ 0 & 2 \end{pmatrix}.$$

In Worked Exercise C67(b) we found that the matrix of t with respect to the eigenvector basis is diagonal and that its diagonal entries are the eigenvalues of the linear transformation t. This is because the matrix of the linear transformation t maps the basis vectors to their images under t, but these basis vectors are precisely the eigenvectors that get mapped to

multiples of themselves. You should find a similar outcome in the next exercise.

Exercise C128

Consider the linear transformation $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$t(x, y) = (x - 2y, -2x - 2y).$$

(a) Write down the matrix of t with respect to the standard basis for $\mathbb{R}^2$.

(b) Find the matrix of t with respect to the eigenvector basis

$$E = \{(-2, 1), (1, 2)\},$$

which you found in Exercise C126.

Worked Exercise C67(b) and Exercise C128(b) are special cases of the following result. We use the letter $\mathbf{D}$ in this result because the matrix is diagonal.

Theorem C59

Let $t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation, let $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an eigenvector basis of t and let $t(\mathbf{e}_j) = \lambda_j \mathbf{e}_j$, for $j = 1, 2, \ldots, n$. Then the matrix of t with respect to the eigenvector basis E is

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Proof Let t and E be as in the statement of the theorem. We use Strategy C19 to find the matrix of t with respect to the eigenvector basis E.

Eigenvector $\mathbf{e}_j$ corresponds to eigenvalue λ_j.

We have

$$t(\mathbf{e}_j) = \lambda_j \mathbf{e}_j, \quad \text{for } j = 1, 2, \ldots, n.$$

We find the E-coordinates of each of these image vectors:

$$\begin{aligned} t(\mathbf{e}_1) &= \lambda_1 \mathbf{e}_1 + 0\mathbf{e}_2 + \cdots + 0\mathbf{e}_n = (\lambda_1, 0, \ldots, 0)_E, \\ t(\mathbf{e}_2) &= 0\mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \cdots + 0\mathbf{e}_n = (0, \lambda_2, \ldots, 0)_E, \\ &\vdots \\ t(\mathbf{e}_n) &= 0\mathbf{e}_1 + 0\mathbf{e}_2 + \cdots + \lambda_n \mathbf{e}_n = (0, 0, \ldots, \lambda_n)_E. \end{aligned}$$

So the matrix of t with respect to the eigenvector basis E is

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

as claimed. ■

Using this result we can easily write down the matrix of a linear transformation with respect to an eigenvector basis.

Exercise C129

Consider the linear transformation $t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by

$$t(x, y, z) = (-x + 2y + 2z, 2x + 2y + 2z, -3x - 6y - 6z),$$

with eigenvector basis

$$E = \{(0, 1, -1), (-2, 1, 0), (1, 0, -1)\}.$$

Use the solution to Exercise C127 to write down the matrix of t with respect to this eigenvector basis.

2.2 Transition matrices

Suppose that $t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear transformation and E is an eigenvector basis of t. We have just shown that the matrix of t with respect to the eigenvector basis E is a diagonal matrix $\mathbf{D}$.

Figures 5 and 6 show the linear transformation t with respect to the eigenvector basis E and the standard basis, respectively.

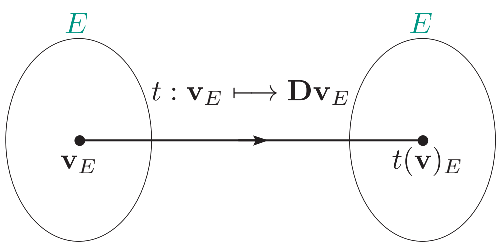


Figure 5 The linear transformation t with eigenvector basis E for the domain and codomain

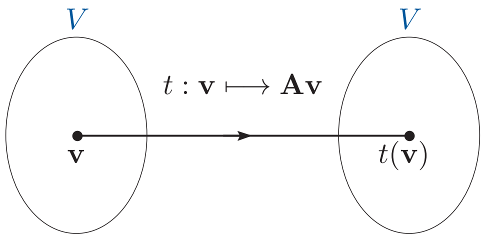


Figure 6 The linear transformation t with standard basis V for the domain and codomain

It is natural to ask whether there is any relationship between this matrix $\mathbf{D}$ and the matrix $\mathbf{A}$ of t with respect to the standard basis for $\mathbb{R}^n$. It turns out that there is an algebraic relationship between the matrices $\mathbf{D}$ and $\mathbf{A}$.

We now show this relationship. To do this, first we find an algebraic relationship between the E-coordinate representation of a vector $\mathbf{v}_E$ (as in Figure 5) and the standard coordinate representation of the same vector (as in Figure 6). We begin by doing this for the example that we considered at the beginning of the section, where $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the linear transformation given by

$$t(x, y) = (x + 3y, 2x - 4y)$$

and E is the eigenvector basis $\{(1, -2), (3, 1)\}$.

Suppose that the E-coordinate representation of a vector $\mathbf{v}$ in $\mathbb{R}^2$ is

$$\mathbf{v}_E = (a, b)_E.$$

What are the standard coordinates of $\mathbf{v}$?

In column form,

$$\mathbf{v} = a\begin{pmatrix} 1 \\ -2 \end{pmatrix} + b\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} a + 3b \\ -2a + b \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}_E.$$

Thus in matrix form we have

$$\mathbf{v} = \mathbf{P}\mathbf{v}_E,$$

where

$$\mathbf{P} = \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}.$$

Now, by the Summary Theorem (Theorem C19 in Unit C1), a square matrix is invertible if and only if its determinant is non-zero. Here we have $\det \mathbf{P} = 1 - (-6) = 7 \neq 0$, so $\mathbf{P}$ is invertible with inverse $\mathbf{P}^{-1}$.

Since $\mathbf{v} = \mathbf{P}\mathbf{v}_E$, it follows that

$$\mathbf{P}^{-1}\mathbf{v} = \mathbf{P}^{-1}(\mathbf{P}\mathbf{v}_E) = (\mathbf{P}^{-1}\mathbf{P})\mathbf{v}_E = \mathbf{v}_E.$$

So multiplication on the left by the matrix $\mathbf{P}$ converts the E-coordinate representation of a vector into the standard coordinate representation and, similarly, multiplication on the left by the matrix $\mathbf{P}^{-1}$ converts the standard coordinate representation of a vector into the E-coordinate representation.

In this case the columns of $\mathbf{P}$ are formed from the standard coordinates of the vectors in E, but this is no coincidence. This simple relationship between the matrix $\mathbf{P}$ and the basis E always holds and we call $\mathbf{P}$ the *transition matrix* from the basis E to the standard basis for $\mathbb{R}^2$.

The general definition is as follows.

Definition

Let $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be a basis for $\mathbb{R}^n$. The **transition matrix P** from the basis E to the standard basis for $\mathbb{R}^n$ is the matrix whose jth column is formed from the standard coordinates of $\mathbf{e}_j$.

Exercise C130

(a) Write down the transition matrix $\mathbf{P}$ from the basis $E = \{(1,3),(2,5)\}$ to the standard basis for $\mathbb{R}^2$.

(b) Write down the transition matrix $\mathbf{P}$ from the basis $E = \{(0,1,-1), (-2,1,0),(1,0,-1)\}$ to the standard basis for $\mathbb{R}^3$.

In the example above, we have seen that the transition matrix $\mathbf{P}$ from the basis $E = \{(1,-2),(3,1)\}$ to the standard basis for $\mathbb{R}^2$ converts E-coordinate representations into standard coordinate representations, and that $\mathbf{P}^{-1}$ converts standard coordinate representations into E-coordinate representations. This is true in general.

Theorem C60

Let $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be a basis for $\mathbb{R}^n$ and let $\mathbf{P}$ be the transition matrix from the basis E to the standard basis for $\mathbb{R}^n$. Then the standard coordinate representation of a vector in $\mathbb{R}^n$ is given by

$$\mathbf{v} = \mathbf{P}\mathbf{v}_E.$$

Moreover, $\mathbf{P}$ is invertible and

$$\mathbf{v}_E = \mathbf{P}^{-1}\mathbf{v}.$$

Proof The matrix $\mathbf{P}$ converts the E-coordinate representation of a vector in $\mathbb{R}^n$ to the standard coordinate representation of the *same* vector in $\mathbb{R}^n$, so in effect it is the matrix of the identity linear transformation $i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with respect to the basis E in the domain and the standard basis in the codomain.

The statement $\mathbf{v} = \mathbf{P}\mathbf{v}_E$ is equivalent to the statement that $\mathbf{P}$ is the matrix of the identity transformation i of $\mathbb{R}^n$ with respect to the basis E for the domain and the standard basis for the codomain.

To find this matrix $\mathbf{P}$, we use Strategy C15 from Unit C3. We begin by finding the images under i of the vectors in the domain basis E:

$$i(\mathbf{e}_1) = \mathbf{e}_1, \quad i(\mathbf{e}_2) = \mathbf{e}_2, \quad \ldots, \quad i(\mathbf{e}_n) = \mathbf{e}_n.$$

It now follows from Strategy C15 that each column of $\mathbf{P}$ is formed from the standard coordinates of the corresponding basis vector, so $\mathbf{P}$ is the transition matrix from the basis E to the standard basis for $\mathbb{R}^n$, as claimed.

We know that the identity transformation i is invertible and that $i^{-1} = i$. It follows from the Inverse Rule (Theorem C45 in Unit C3) that $\mathbf{P}$ is invertible and that $\mathbf{P}^{-1}$ is the matrix of $i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with respect to the standard basis for the domain and the basis E for the codomain; that is,

$$\mathbf{v} \longmapsto \mathbf{v}_E = \mathbf{P}^{-1}\mathbf{v}.$$ ■

When E is the standard basis for $\mathbb{R}^n$, the matrix $\mathbf{P}$ is the identity matrix $\mathbf{I}_n$, as you would expect.

We also get the following corollary from Theorem C60.

Corollary C61

The rows or columns of an $n \times n$ matrix $\mathbf{A}$ form a set of n linearly independent vectors if and only if $\det \mathbf{A} \neq 0$.

Proof Let $\mathbf{A}$ be an $n \times n$ matrix.

We start by proving the *only if* part.

We first show that if the columns of $\mathbf{A}$ are linearly independent, then $\det \mathbf{A} \neq 0$.

Suppose the columns are linearly independent, then the columns form a basis for $\mathbb{R}^n$ and $\mathbf{A}$ is the transition matrix from this basis to the standard basis. Hence $\mathbf{A}$ is invertible by Theorem C60, and so $\det \mathbf{A} \neq 0$ by the Summary Theorem (Theorem C19 in Unit C1).

If the rows of $\mathbf{A}$ are linearly independent then we consider the transpose $\mathbf{A}^T$.

Suppose the rows of $\mathbf{A}$ are linearly independent, then the columns of $\mathbf{A}^T$ are linearly independent and $\det \mathbf{A}^T \neq 0$ by the above reasoning. We have $\det \mathbf{A} = \det \mathbf{A}^T$ by Theorem C14 in Unit C1, and hence $\det \mathbf{A} \neq 0$, as required.

We now prove the *if* part using the contrapositive; that is, we show that if the rows or columns of $\mathbf{A}$ are *not* linearly independent then $\det \mathbf{A} = 0$.

Suppose the rows of $\mathbf{A}$ form a linearly dependent set, then the row-reduced form of $\mathbf{A}$ contains a zero row, so $\mathbf{A}$ is not invertible by the Invertibility Theorem (Theorem C7 in Unit C1), and hence $\det \mathbf{A} = 0$ by the Summary Theorem.

Suppose the columns of $\mathbf{A}$ form a linearly dependent set, then the rows of $\mathbf{A}^T$ are linearly dependent and $\det \mathbf{A} = \det \mathbf{A}^T = 0$ by the above reasoning.

Hence, if $\det \mathbf{A} \neq 0$, then the rows or columns of $\mathbf{A}$ form a linearly independent set of vectors. ■

Recall that our aim in this subsection is to relate the matrices $\mathbf{D}$ and $\mathbf{A}$, where $\mathbf{D}$ is the matrix of a linear transformation $t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with respect to an eigenvector basis of t, and $\mathbf{A}$ is the matrix of t with respect to the standard basis for $\mathbb{R}^n$. Figure 7 shows how we can do this by using the transition matrix $\mathbf{P}$ from the eigenvector basis E to the standard basis for $\mathbb{R}^n$, so linking together Figures 5 and 6.

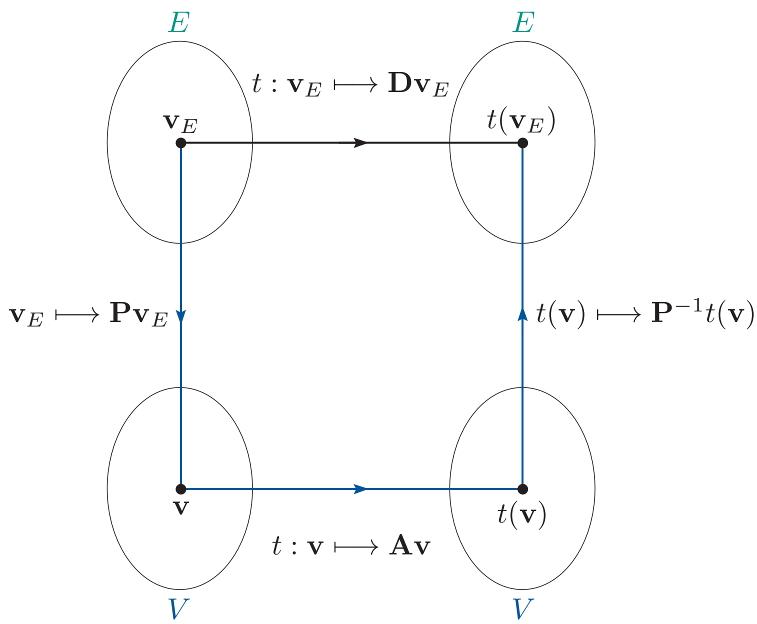


Figure 7 The transition matrix $\mathbf{P}$ from the eigenvector basis E of t to the standard basis for $\mathbb{R}^n$

The top line of the diagram shows that multiplication by $\mathbf{D}$ converts the E-coordinate representation of $\mathbf{v}$ to the E-coordinate representation of $t(\mathbf{v})$:

$$t(\mathbf{v})_E = \mathbf{D}\mathbf{v}_E. \tag{3}$$

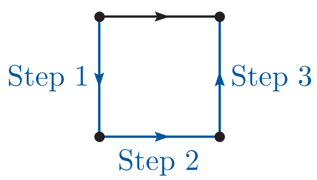


Figure 8 The transition in three steps

The diagram also shows that this change can be achieved in another way, in three steps, highlighted in Figure 8.

1. Use the transition matrix $\mathbf{P}$ to convert the E-coordinate representation of $\mathbf{v}$ to the standard coordinate representation of $\mathbf{v}$:

$$\mathbf{v} = \mathbf{P}\mathbf{v}_E.$$

2. Multiply $\mathbf{v}$ on the left by matrix $\mathbf{A}$ to obtain the standard coordinate representation of $t(\mathbf{v})$:

$$t(\mathbf{v}) = \mathbf{A}\mathbf{v} = \mathbf{A}\mathbf{P}\mathbf{v}_E.$$

3. Use the matrix $\mathbf{P}^{-1}$ to convert the standard coordinate representation of $t(\mathbf{v})$ to the E-coordinate representation of $t(\mathbf{v})$:

$$t(\mathbf{v})_E = \mathbf{P}^{-1}t(\mathbf{v}) = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{v}_E.$$

Comparing this last equation with equation (3), we see that $\mathbf{D}$, $\mathbf{A}$ and $\mathbf{P}$ are related by the equation

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}.$$

Thus we have proved the following result.

Theorem C62

Let $t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation and let E be an eigenvector basis of t. Let $\mathbf{A}$ be the matrix of t with respect to the standard basis for $\mathbb{R}^n$, let $\mathbf{D}$ be the matrix of t with respect to the eigenvector basis E and let $\mathbf{P}$ be the transition matrix from E to the standard basis for $\mathbb{R}^n$. Then

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}.$$

In fact, Theorem C62 holds for *any* basis E for $\mathbb{R}^n$, although $\mathbf{D}$ is diagonal only when E is an eigenvector basis.

Since $\mathbf{D}$, $\mathbf{A}$, $\mathbf{P}$ and $\mathbf{P}^{-1}$ are all square $n \times n$ matrices, we can multiply $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ on the left by the matrix $\mathbf{P}$ and on the right by the matrix $\mathbf{P}^{-1}$ to obtain the related equation

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}.$$

This algebraic relationship $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ may remind you of the algebraic relationship

$$y = g \circ x \circ g^{-1}$$

between *conjugate* permutations x and y in the symmetric group S_n, which you met in Subsection 4.1 of Unit B3. You saw in Unit C1 that the set of square invertible $n \times n$ matrices form a group under multiplication, and here the change of basis is in some sense equivalent to the 'renaming' in permutations. The matrices $\mathbf{D}$ and $\mathbf{A}$ are *conjugate matrices*: we will not use this concept here, but you will meet this idea of conjugacy in groups again in Book E.

We end this subsection by applying Theorem C62 to some examples.

Consider the linear transformation $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$t(x, y) = (x + 3y, 2x - 4y).$$

In Worked Exercise C67 you saw that

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}$$

is the matrix of t with respect to the standard basis for $\mathbb{R}^2$ and that

$$\mathbf{D} = \begin{pmatrix} -5 & 0 \\ 0 & 2 \end{pmatrix}$$

is the matrix of t with respect to the eigenvector basis $E = \{(1,-2),(3,1)\}$.

At the beginning of this subsection you saw that the transition matrix from the basis E to the standard basis for $\mathbb{R}^2$ is

$$\mathbf{P} = \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}.$$

Now, using Strategy C4 from Unit C1, we have

$$\mathbf{P}^{-1} = \frac{1}{7}\begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{7} & -\frac{3}{7} \\ \frac{2}{7} & \frac{1}{7} \end{pmatrix},$$

so

$$\begin{aligned} \mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \begin{pmatrix} \frac{1}{7} & -\frac{3}{7} \\ \frac{2}{7} & \frac{1}{7} \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -5 & 0 \\ 0 & 2 \end{pmatrix} \\ &= \mathbf{D}, \end{aligned}$$

as claimed.

Exercise C131

Use the solution to Exercise C128 to find a matrix $\mathbf{P}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -2 & -2 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}.$$

2.3 Diagonalisation

In this subsection you will consider the problem of determining when a matrix is *diagonalisable* and how to *diagonalise* a matrix when it is possible.

Definition

The matrix $\mathbf{A}$ is **diagonalisable** if there exists an invertible matrix $\mathbf{P}$ such that the matrix

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

is diagonal.

Clearly the matrices $\mathbf{A}$, $\mathbf{D}$ and $\mathbf{P}$ must all be square matrices of the same size.

If a matrix $\mathbf{A}$ is diagonalisable, then to diagonalise it we need to find both the diagonal matrix $\mathbf{D}$ *and* the invertible matrix $\mathbf{P}$, since it is this transition matrix $\mathbf{P}$ that links the matrix $\mathbf{A}$ with the diagonal matrix $\mathbf{D}$.

One particular use of diagonalisation of matrices is to find powers of matrices. We saw earlier that multiplying $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ on the left by $\mathbf{P}$ and on the right by $\mathbf{P}^{-1}$ gives $\mathbf{A} = \mathbf{PDP}^{-1}$. Now consider powers of $\mathbf{A}$,

$$\begin{aligned}\mathbf{A}^2 &= (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})\\ &= \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P})\mathbf{DP}^{-1}\\ &= \mathbf{PDDP}^{-1}\\ &= \mathbf{PD}^2\mathbf{P}^{-1},\end{aligned}$$

and, in general we have

$$\mathbf{A}^n = \mathbf{PD}^n\mathbf{P}^{-1}, \quad \text{for } n = 1, 2, \ldots.$$

This last equation is useful for calculating powers of matrices, since calculating the nth power of a diagonal matrix is particularly simple: you need to find only the nth power of each diagonal entry. But first we need to be able to find both $\mathbf{D}$ and $\mathbf{P}$ (from which we can find $\mathbf{P}^{-1}$).

Exercise C132

(a) Write down $\mathbf{D}^5$, where $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$.

(b) Calculate $\mathbf{A}^5$, where $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -2 & -2 \end{pmatrix}$.

(In Exercise C131 you found that $\mathbf{P} = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$ satisfies $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$.)

If $\mathbf{A}$ is any $n \times n$ matrix, then we can define a linear transformation t as:

$$\begin{aligned} t : \mathbb{R}^n &\longrightarrow \mathbb{R}^n\\ \mathbf{v} &\longmapsto \mathbf{Av}.\end{aligned}$$

In Section 1 we said that $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ with corresponding eigenvalue λ if $\mathbf{Av} = t(\mathbf{v}) = \lambda\mathbf{v}$; that is, if $\mathbf{v}$ is an eigenvector of t.

Definition

Let $\mathbf{A}$ be an $n \times n$ matrix and let $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be a basis for $\mathbb{R}^n$ consisting of eigenvectors of $\mathbf{A}$. The basis E is an **eigenvector basis** of $\mathbf{A}$.

Thus E is an eigenvector basis of $\mathbf{A}$ if E is an eigenvector basis of t.

Worked Exercise C68

Find an eigenvector basis of the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}.$$

Solution

We found the eigenvectors of $\mathbf{A}$ in Worked Exercise C63.

The eigenvectors of $\mathbf{A}$ are the non-zero vectors of the following forms:

(k, k), corresponding to the eigenvalue $\lambda = 7$,

$(k, -k)$, corresponding to the eigenvalue $\lambda = 3$.

Since $(1, 1)$ and $(1, -1)$ are eigenvectors of $\mathbf{A}$, and $(1, -1)$ is not a multiple of $(1, 1)$, the set $E = \{(1, 1), (1, -1)\}$ is an eigenvector basis of $\mathbf{A}$.

Suppose that E is an eigenvector basis of the $n \times n$ matrix $\mathbf{A}$; that is, E is an eigenvector basis of the linear transformation $t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ given by

$$t(\mathbf{v}) = \mathbf{A}\mathbf{v}.$$

It follows from Theorems C59 and C62 that if $\mathbf{P}$ is the transition matrix from the basis E to the standard basis for $\mathbb{R}^n$, then

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

is diagonal; that is, $\mathbf{A}$ is diagonalisable. This gives the following strategy for diagonalising a matrix, when this is possible.

Strategy C20

To diagonalise an $n \times n$ matrix $\mathbf{A}$:

1. find all the eigenvalues of $\mathbf{A}$
2. find (if possible) an eigenvector basis $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of $\mathbf{A}$
3. write down the transition matrix $\mathbf{P}$ whose jth column is formed from the standard coordinates of $\mathbf{e}_j$.

Then

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where λ_j is the eigenvalue corresponding to the eigenvector $\mathbf{e}_j$.

The order of the eigenvalues down the diagonal of $\mathbf{D}$ must match the order of the eigenvectors in the basis E used to construct the transition matrix $\mathbf{P}$. When asked to diagonalise a matrix, it is not enough to write down a diagonal matrix containing the eigenvalues: you must also give the transition matrix $\mathbf{P}$.

The complexity involved in finding an eigenvector basis of $\mathbf{A}$ in step 2 of Strategy C20 depends on the matrix $\mathbf{A}$. In Worked Exercise C68 we formed an eigenvector basis of $\mathbf{A}$ by taking one eigenvector corresponding to each eigenvalue, ensuring that the eigenvectors were linearly independent. In general, we have the following result, which we will prove at the end of this subsection after looking at how it can be used. This result means that any eigenvector can be chosen for each (distinct) eigenvalue and there is no need to check that they are linearly independent.

Theorem C63

Let $\mathbf{A}$ be an $n \times n$ matrix with *distinct* eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. Then $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is an eigenvector basis of $\mathbf{A}$.

We give an example of how Theorem C63 can be used.

Worked Exercise C69

Diagonalise the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix}.$$

Solution

We use Strategy C20.

We found the eigenvalues and eigenvectors of $\mathbf{A}$ in Worked Exercise C64.

The eigenvalues of $\mathbf{A}$ are $\lambda = 3$, $\lambda = 2$ and $\lambda = 1$.

The eigenvectors of $\mathbf{A}$ are the non-zero vectors of the following forms:

$$(k, 2k, k), \text{ corresponding to } \lambda = 3,$$
$$(0, k, 0), \text{ corresponding to } \lambda = 2,$$
$$(k, 4k, -k), \text{ corresponding to } \lambda = 1.$$

It follows from Theorem C63 that we can form an eigenvector basis of $\mathbf{A}$ by taking one eigenvector corresponding to each of the three distinct eigenvalues. For example,

$$E = \{(1,2,1), (0,1,0), (1,4,-1)\}$$

is an eigenvector basis of $\mathbf{A}$.

We use the eigenvectors in E to form the columns of the transition matrix:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 1 & 0 & -1 \end{pmatrix}.$$

Remember that the eigenvalues in $\mathbf{D}$ must appear in the same order as the corresponding eigenvectors in $\mathbf{P}$.

We use the eigenvalues corresponding to the eigenvectors in E to form the diagonal matrix:

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If the eigenvectors had been chosen in a different order, then the order of the columns of the transition matrix $\mathbf{P}$ and the order of the diagonal entries of the resulting matrix $\mathbf{D}$ would have been different.

In addition, other transition matrices arise from using different eigenvectors for the eigenvector basis.

Another solution is

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \text{where} \quad \mathbf{P} = \begin{pmatrix} 0 & -1 & 1 \\ 2 & -4 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

Both the order of the eigenvalues, and the eigenvectors chosen for the columns of P, differ here.

Exercise C133

Diagonalise the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 2 \end{pmatrix}.$$

(In Exercise C120 you found that the eigenvectors of $\mathbf{A}$ are the non-zero vectors $(2k, 2k, k)$, $(-2k, k, 2k)$ and $(k, -2k, 2k)$, corresponding to the eigenvalues $\lambda = 6$, $\lambda = 3$ and $\lambda = 0$, respectively.)

It may be possible to find an eigenvector basis of an $n \times n$ matrix $\mathbf{A}$ even when $\mathbf{A}$ does not have n distinct eigenvalues.

Strategy C21

To find an eigenvector basis of an $n \times n$ matrix $\mathbf{A}$:

1. find a basis for each eigenspace of $\mathbf{A}$
2. form the set E of all the basis vectors found in step 1.

If there are n vectors in E, then E is an eigenvector basis of $\mathbf{A}$; otherwise E is not a basis.

The fact that E, as found in Strategy C21, is an eigenvector basis of $\mathbf{A}$ if and only if there are n vectors in E, can be proved in a similar way to Theorem C63, but the details are more complicated.

Worked Exercise C70

Diagonalise the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

given that the eigenvalues of $\mathbf{A}$ are $\lambda = 8$, $\lambda = 2$ and $\lambda = 2$.

Solution

There are many possible solutions to this, and to each of the remaining exercises in this section.

We use Strategies C20 and C21, but start with the second part of Strategy C18. In general we will not list all the strategies involved.

To find the eigenspaces of $\mathbf{A}$, we consider the eigenvector equations

$$\begin{aligned} (4-\lambda)x + \quad 2y + \quad 2z &= 0 \\ 2x + (4-\lambda)y + \quad 2z &= 0 \\ 2x + \quad 2y + (4-\lambda)z &= 0, \end{aligned}$$

for each eigenvalue.

$\boxed{\lambda = 8}$ The eigenvector equations become

$$\begin{aligned} -4x + 2y + 2z &= 0 \\ 2x - 4y + 2z &= 0 \\ 2x + 2y - 4z &= 0. \end{aligned}$$

Subtracting the second equation from the first, we obtain $-6x + 6y = 0$, which implies that $x = y$. Substituting this into the third equation, we obtain $4x - 4z = 0$, which implies that $x = z$.

Thus $S(8) = \{(k, k, k) : k \in \mathbb{R}\}$.

$\boxed{\lambda = 2}$ All three eigenvector equations become

$$2x + 2y + 2z = 0,$$

that is, $x + y + z = 0$, so $z = -(x + y)$.

Thus $S(2) = \{(k, l, -(k + l)) : k, l \in \mathbb{R}\}$.

Any vector in $S(8)$ can be written as $k(1, 1, 1)$, and any vector in $S(2)$ can be written as $k(1, 0, -1) + l(0, 1, -1)$.

A basis for $S(8)$ is $\{(1, 1, 1)\}$ and a basis for $S(2)$ is $\{(1, 0, -1), (0, 1, -1)\}$. The set

$$E = \{(1, 1, 1), (1, 0, -1), (0, 1, -1)\}$$

contains three vectors, so it is an eigenvector basis of $\mathbf{A}$.

Note that Strategy C21 does not require us to prove linear independence of the vectors in E: combining the bases of the eigenspaces $S(2)$ and $S(8)$ gives a set of linearly independent vectors.

We use the eigenvectors in E to form the columns of the transition matrix:

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}.$$

We use the eigenvalues corresponding to the eigenvectors in E to form the diagonal matrix:

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Exercise C134

Diagonalise the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

If the matrix $\mathbf{A}$ does *not* have an eigenvector basis, then these methods cannot be applied and the matrix $\mathbf{A}$ is not diagonalisable – there is no transition matrix. For example, in Exercise C124 you saw that all the eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

are non-zero vectors of the form $(k, 0)$. Any two eigenvectors of $\mathbf{A}$ are linearly dependent, so there is no eigenvector basis. Thus there is no transition matrix and $\mathbf{A}$ is not diagonalisable.

Similarly the matrix

$$\mathbf{B} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 4 & 1 \\ -1 & 1 & 4 \end{pmatrix}$$

from Exercise C125 is also not diagonalisable. The eigenvectors corresponding to the eigenvalue $\lambda = 2$ of multiplicity 2 are the non-zero vectors of the form $(k, -k, k)$, so any two eigenvectors of $\mathbf{B}$ in $S(2)$ are linearly dependent. The other eigenvalue $\lambda = 5$ has multiplicity 1. As stated at the end of Section 1, the dimension of an eigenspace cannot exceed the multiplicity of the corresponding eigenvalue, and so there cannot be two linearly independent eigenvectors corresponding to eigenvalue $\lambda = 5$.

Therefore there is no set of three linearly independent eigenvectors and thus no eigenvector basis; there is no transition matrix and thus $\mathbf{B}$ is not diagonalisable.

We have shown that, if the matrix $\mathbf{A}$ of a linear transformation t has an eigenvector basis, then using this basis for both the domain and codomain results in a matrix of t that is a diagonal matrix. On the other hand, if there is an eigenvalue of multiplicity m for which there are fewer than m linearly independent eigenvectors, then there is no eigenvector basis and matrix $\mathbf{A}$ is not diagonalisable.

We end this section by proving Theorem C63 as promised.

Theorem C63

Let $\mathbf{A}$ be an $n \times n$ matrix with *distinct* eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. Then $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is an eigenvector basis of $\mathbf{A}$.

Proof Let $\mathbf{A}$ and E be as in the statement of the theorem.

Since any linearly independent set of n vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$, by Theorem C25 in Unit C2, we need show only that E is linearly independent. To do this, we assume that E is linearly dependent and obtain a contradiction.

If E is linearly independent, then E must be an eigenvector basis of $\mathbf{A}$.

Suppose to the contrary that E is linearly dependent. Then we can take the smallest value of m $(2 \leq m \leq n)$ for which a set of m vectors in E is linearly dependent. By relabelling the eigenvectors (if necessary), we can write

$$\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \cdots + \alpha_m \mathbf{e}_m = \mathbf{0}, \tag{4}$$

with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, ..., $\alpha_m \neq 0$.

Multiplying both sides of equation (4) by matrix $\mathbf{A}$, we obtain

$$\mathbf{A}(\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \cdots + \alpha_m \mathbf{e}_m) = \mathbf{A0},$$

that is,

$$\alpha_1 \mathbf{A}\mathbf{e}_1 + \alpha_2 \mathbf{A}\mathbf{e}_2 + \cdots + \alpha_m \mathbf{A}\mathbf{e}_m = \mathbf{0}.$$

Now, $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ are eigenvectors of $\mathbf{A}$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, so $\mathbf{A}\mathbf{e}_j = \lambda_j \mathbf{e}_j$ and

$$\alpha_1 \lambda_1 \mathbf{e}_1 + \alpha_2 \lambda_2 \mathbf{e}_2 + \cdots + \alpha_m \lambda_m \mathbf{e}_m = \mathbf{0}. \tag{5}$$

We now eliminate the vector $\mathbf{e}_m$. To do this, we multiply equation (4) by λ_m and subtract the result from equation (5):

$$\alpha_1(\lambda_1 - \lambda_m)\mathbf{e}_1 + \alpha_2(\lambda_2 - \lambda_m)\mathbf{e}_2 + \cdots + \alpha_{m-1}(\lambda_{m-1} - \lambda_m)\mathbf{e}_{m-1} = \mathbf{0}.$$

Since the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct, and none of the numbers $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}$ is zero, we deduce that the set of $m - 1$ vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{m-1}\}$ is linearly dependent. This, however, is impossible since we assumed that m is the *smallest* number such that a set of m vectors in E is linearly dependent. This contradiction establishes the result. ■

3 Symmetric matrices

In this section you will concentrate on diagonalising symmetric matrices. You will see that such matrices are always diagonalisable and that their transition matrices can be chosen to have particular properties.

3.1 Diagonalising symmetric matrices

Suppose that $\mathbf{A}$ is an $n \times n$ matrix and that we can find a basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for $\mathbb{R}^n$ consisting of eigenvectors of $\mathbf{A}$. In Section 2 you saw that $\mathbf{A}$ can be diagonalised: if $\mathbf{P}$ is the transition matrix whose columns are formed from the coordinates of the eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, then

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

is a diagonal matrix.

In this section you will see that whenever $\mathbf{A}$ is an $n \times n$ *symmetric* matrix (a matrix where $\mathbf{A}^T = \mathbf{A}$), then we can always find a basis for $\mathbb{R}^n$ made up of eigenvectors of $\mathbf{A}$, and so such a matrix is always diagonalisable. In fact, we can always find an *orthonormal* basis for $\mathbb{R}^n$ made up of eigenvectors of $\mathbf{A}$. Recall from Subsection 5.4 of Unit C2 that an *orthonormal* basis

consists of mutually perpendicular (*orthogonal*) vectors of magnitude 1. For example, the standard basis for $\mathbb{R}^n$ is an orthonormal basis.

When we have an orthonormal basis, it turns out that the inverse of the transition matrix $\mathbf{P}$ is actually the transpose of $\mathbf{P}$; that is, $\mathbf{P}^{-1} = \mathbf{P}^T$. This can be useful since finding the transpose of a matrix is much simpler than finding the inverse. We will prove this result as Theorem C65 in the next subsection where you will also see that orthogonal matrices have other useful properties.

For example, consider the symmetric matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 2 \end{pmatrix}.$$

We will show that there is an orthonormal basis for $\mathbb{R}^3$ that consists of eigenvectors of $\mathbf{A}$.

You found in Exercise C120 that the eigenvalues of $\mathbf{A}$ are $\lambda = 6$, $\lambda = 3$ and $\lambda = 0$, and that the eigenvectors are the non-zero vectors of the following forms:

$$\begin{aligned} &(2k, 2k, k), \text{ corresponding to } \lambda = 6, \\ &(-2k, k, 2k), \text{ corresponding to } \lambda = 3, \\ &(k, -2k, 2k), \text{ corresponding to } \lambda = 0. \end{aligned}$$

Exercise C135

Let $\mathbf{v}_1 = (2k, 2k, k)$, $\mathbf{v}_2 = (-2l, l, 2l)$ and $\mathbf{v}_3 = (m, -2m, 2m)$, where k, l, m are positive real numbers.

(a) Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an *orthogonal* basis for $\mathbb{R}^3$.

(b) Find values of k, l and m for which $|\mathbf{v}_1| = |\mathbf{v}_2| = |\mathbf{v}_3| = 1$.

In Subsection 5.4 of Unit C2 you saw that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$ if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$, and $|\mathbf{v}_i| = 1$ for each i. It follows from Exercise C135 that

$$E = \left\{ \left(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}\right), \left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}\right), \left(\tfrac{1}{3}, -\tfrac{2}{3}, \tfrac{2}{3}\right) \right\}$$

is an orthonormal basis for $\mathbb{R}^3$. Since E is an eigenvector basis of $\mathbf{A}$, we say that E is an *orthonormal eigenvector basis* of $\mathbf{A}$.

Following Strategy C20, we diagonalise the matrix $\mathbf{A}$ by writing down the transition matrix $\mathbf{P}$ whose columns are formed from the standard coordinates of the vectors in E:

$$\mathbf{P} = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}.$$

A transition matrix formed from an orthonormal eigenvector basis in this way is called an *orthogonal* matrix.

Definition

An $n \times n$ matrix whose columns form an orthonormal basis for $\mathbb{R}^n$ is an **orthogonal** matrix.

It is important to remember that the columns of an orthogonal matrix are *orthonormal* vectors, not just *orthogonal* vectors, despite the name!

Consider the 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

The columns of $\mathbf{A}$ (as vectors) are orthogonal since

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 0.$$

Orthogonal vectors are linearly independent, so the columns of $\mathbf{A}$ form a basis for $\mathbb{R}^2$.

The columns of $\mathbf{A}$ (as vectors) also have magnitude 1 since

$$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1, \quad \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = 1,$$

so the matrix $\mathbf{A}$ is an orthogonal matrix.

Exercise C136

Show that $\mathbf{P}^T\mathbf{P} = \mathbf{I}$, where

$$\mathbf{P} = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}.$$

($\mathbf{P}$ is the orthogonal matrix formed below Exercise C135.)

We know that if $\mathbf{P}^T\mathbf{P} = \mathbf{I}$, then $\mathbf{P}\mathbf{P}^T = \mathbf{I}$ (by Theorem C18 in Unit C1), so for the matrix $\mathbf{P}$ in Exercise C136, $\mathbf{P}^T$ is the inverse of $\mathbf{P}$; that is, $\mathbf{P}^T = \mathbf{P}^{-1}$. We will prove that $\mathbf{P}^T = \mathbf{P}^{-1}$ for *any* orthogonal matrix $\mathbf{P}$ as Theorem C65 in the next subsection.

It follows from this and Strategy C20 that

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We say that the matrix $\mathbf{A}$ has been *orthogonally diagonalised*.

Definition

The matrix $\mathbf{A}$ is **orthogonally diagonalisable** if there exists an orthogonal matrix $\mathbf{P}$ such that the matrix

$$\mathbf{D} = \mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

is diagonal.

The following strategy is a modification of Strategy C20 for diagonalising a matrix.

Strategy C22

To orthogonally diagonalise an $n \times n$ symmetric matrix $\mathbf{A}$:

1. find all the eigenvalues of $\mathbf{A}$
2. find an orthonormal eigenvector basis $E = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of $\mathbf{A}$
3. write down the orthogonal transition matrix $\mathbf{P}$ whose jth column is formed from the standard coordinates of $\mathbf{e}_j$.

Then

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where λ_j is the eigenvalue corresponding to the eigenvector $\mathbf{e}_j$.

In Section 4 you will see that orthogonal diagonalisation is used for classifying conics and quadrics. However, if the aim is simply to *diagonalise* a symmetric matrix as opposed to *orthogonally diagonalise* it, then use Strategy C20 – this saves time and effort when an orthonormal basis, or equivalently an orthogonal transition matrix, is not required. It is always a good idea to consider carefully what a problem requires you to do in order to solve it in the most efficient way.

You may have noticed that the words ‘if possible’ appear in Strategy C20, but not in Strategy C22. This is due to the fact that an $n \times n$ symmetric matrix $\mathbf{A}$ *always* has an orthonormal eigenvector basis, so it must be orthogonally diagonalisable. It is also true that any orthogonally diagonalisable matrix $\mathbf{A}$ must be symmetric – you might like to prove this yourself; it is included as a ‘challenging’ exercise in the additional exercises booklet for this unit.

In the case where a symmetric matrix $\mathbf{A}$ has n distinct eigenvalues, the fact that $\mathbf{A}$ has an orthonormal eigenvector basis follows from the following result.

Theorem C64

Eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal.

Proof Let $\mathbf{A}$ be a symmetric matrix, and let $\mathbf{v}$ and $\mathbf{w}$ be eigenvectors of $\mathbf{A}$ corresponding to the distinct eigenvalues λ and μ. Then

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \quad \text{and} \quad \mathbf{A}\mathbf{w} = \mu\mathbf{w}.$$

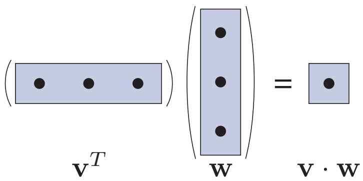


Figure 9 $\mathbf{v}^T\mathbf{w} = \mathbf{v} \cdot \mathbf{w}$

To show that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, we need to show that $\mathbf{v} \cdot \mathbf{w} = 0$. We do this by writing $\mathbf{v}^T\mathbf{A}\mathbf{w}$ in two ways and using the fact that $\mathbf{v}^T\mathbf{w} = \mathbf{v} \cdot \mathbf{w}$. This fact is illustrated in Figure 9.

We have,

$$\mathbf{v}^T\mathbf{A}\mathbf{w} = \mathbf{v}^T(\mathbf{A}\mathbf{w}) = \mathbf{v}^T(\mu\mathbf{w}) = \mu(\mathbf{v}^T\mathbf{w}) = \mu(\mathbf{v} \cdot \mathbf{w}).$$

Since $\mathbf{A}$ is symmetric, we have $\mathbf{A}^T = \mathbf{A}$, and therefore that

$$\mathbf{v}^T\mathbf{A} = \mathbf{v}^T\mathbf{A}^T = (\mathbf{A}\mathbf{v})^T.$$

It follows that

$$\mathbf{v}^T\mathbf{A}\mathbf{w} = (\mathbf{v}^T\mathbf{A})\mathbf{w} = (\mathbf{A}\mathbf{v})^T\mathbf{w} = (\lambda\mathbf{v})^T\mathbf{w} = \lambda(\mathbf{v}^T\mathbf{w}) = \lambda(\mathbf{v} \cdot \mathbf{w}).$$

Therefore $\lambda(\mathbf{v} \cdot \mathbf{w}) = \mu(\mathbf{v} \cdot \mathbf{w})$; thus

$$(\lambda - \mu)(\mathbf{v} \cdot \mathbf{w}) = 0.$$

Since the eigenvalues λ and μ are distinct, $\lambda - \mu$ is non-zero, and hence $\mathbf{v} \cdot \mathbf{w} = 0$. The two eigenvectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal as required. ■

The following exercises show how Theorem C64 can be used.

Worked Exercise C71

Orthogonally diagonalise the symmetric matrix

$$\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}.$$

Solution

We use Strategy C22.

We found the eigenvalues and eigenvectors of $\mathbf{A}$ in Worked Exercise C63.

The eigenvalues of $\mathbf{A}$ are $\lambda = 7$ and $\lambda = 3$.

The eigenvectors of $\mathbf{A}$ are the non-zero vectors of the following form:

$$(k, k), \text{ corresponding to } \lambda = 7,$$
$$(k, -k), \text{ corresponding to } \lambda = 3.$$

Since any eigenvectors corresponding to these eigenvalues are orthogonal by Theorem C64, we form an orthonormal eigenvector basis of $\mathbf{A}$ by taking an eigenvector of magnitude 1 corresponding to each of the two distinct eigenvalues.

An eigenvector of magnitude 1 corresponding to $\lambda = 7$ is

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

An eigenvector of magnitude 1 corresponding to $\lambda = 3$ is

$$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

It follows from Theorem C64 that an orthonormal eigenvector basis of $\mathbf{A}$ is

$$E = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\}.$$

We use the eigenvectors in E to form the columns of the orthogonal transition matrix:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

We use the eigenvalues corresponding to the eigenvectors in E to form the diagonal matrix:

$$\mathbf{P}^T\mathbf{AP} = \mathbf{D} = \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix}.$$

Exercise C137

Orthogonally diagonalise each of the following symmetric matrices.

(a) $\mathbf{A} = \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$

The eigenvalues in part (b) are $\lambda = 6$, $\lambda = 3$ and $\lambda = 2$.

So far, in each case where we have orthogonally diagonalised an $n \times n$ symmetric matrix, we have had n distinct eigenvalues and Theorem C64 has ensured that the eigenvectors are all orthogonal. We have then formed an orthonormal eigenvector basis for the matrix by writing down basis vectors of magnitude 1. Where the eigenvalues of the symmetric matrix are not all distinct we have to find an orthonormal eigenvector basis for each eigenspace – then Theorem C64 will ensure that the resulting set of eigenvectors will form an orthonormal eigenvector basis for the matrix.

The following strategy is a modification of Strategy C21. It reflects the fact that we can always find an orthonormal basis comprising r vectors for an eigenspace of a *symmetric* matrix corresponding to an eigenvalue of multiplicity r. This result is not proved here.

Strategy C23

To find an orthonormal eigenvector basis of a *symmetric* matrix $\mathbf{A}$:

1. find an orthonormal basis for each eigenspace of $\mathbf{A}$
2. form the set E of all the basis vectors found in step 1.

Then E is an orthonormal eigenvector basis of $\mathbf{A}$.

Worked Exercise C72

Orthogonally diagonalise the symmetric matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

Solution

In Worked Exercise C70 we found an eigenvector basis of $\mathbf{A}$: $E = \{(1,1,1), (1,0,-1), (0,1,-1)\}$ corresponding to the eigenvalues $\lambda = 8$, $\lambda = 2$ and $\lambda = 2$, respectively.

A basis for the eigenspace $S(8)$ is $\{(1,1,1)\}$, so an orthonormal basis for $S(8)$ is

$$\left\{ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \right\}.$$

A basis for the eigenspace $S(2)$ is $\{(1,0,-1), (0,1,-1)\}$.

These two basis vectors are not orthogonal since

$$(1,0,-1) \cdot (0,1,-1) = 1 \neq 0.$$

We need to find a pair of orthogonal vectors that span $S(2)$. One method for this is the Gram–Schmidt orthogonalisation process that you met in Subsection 5.3 of Unit C2.

To find an *orthogonal* basis for the eigenspace $S(2)$, we use the Gram–Schmidt orthogonalisation process.

Let the orthogonal basis we seek be $\{\mathbf{v}_1, \mathbf{v}_2\}$, with $\mathbf{v}_1 = (1, 0, -1)$.

Then

$$\begin{aligned}\mathbf{v}_2 &= (0,1,-1) - \left(\frac{\mathbf{v}_1 \cdot (0,1,-1)}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 \\ &= (0,1,-1) - \left(\frac{(1,0,-1)\cdot(0,1,-1)}{(1,0,-1)\cdot(1,0,-1)}\right)(1,0,-1) \\ &= (0,1,-1) - \tfrac{1}{2}(1,0,-1) \\ &= \left(-\tfrac{1}{2}, 1, -\tfrac{1}{2}\right).\end{aligned}$$

Dividing $\mathbf{v}_2$ by $|\mathbf{v}_2| = \sqrt{6}/2$ gives a unit basis vector. However, although it is not necessary it is often helpful to minimise the minus signs involved: we can multiply through by -1 to get another unit basis vector orthogonal to $\mathbf{v}_1$.

An orthonormal basis for $S(2)$ is therefore

$$\left\{\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)\right\}.$$

We have ensured that the eigenvectors in the basis for $S(2)$ are orthogonal, and by Theorem C64 the eigenvectors corresponding to the distinct eigenvalues $\lambda = 8$ and $\lambda = 2$ are orthogonal.

By Theorem C64 an orthonormal eigenvector basis of $\mathbf{A}$ is therefore

$$E = \left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)\right\}.$$

We use the eigenvectors in E to form the columns of the transition matrix:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}.$$

We use the eigenvalues corresponding to the eigenvectors in E to form the diagonal matrix:

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The diagonal matrix found here is the same as that found in Worked Exercise C70, since the eigenvalues are considered in the same order. The difference in the diagonalisation lies in the transition matrix, which in this case is orthogonal.

Exercise C138

Orthogonally diagonalise the symmetric matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

(In Exercise C134 you found the eigenvalues and eigenvectors of $\mathbf{A}$: that a basis for $S(3)$ is $\{(0,1,1)\}$ and a basis for $S(1)$ is $\{(1,0,0),(0,1,-1)\}$.)

We conclude this subsection by noting that every symmetric matrix can be orthogonally diagonalised and conversely that an orthogonally diagonalisable matrix is symmetric. However, it is possible to diagonalise (but not orthogonally diagonalise) a non-symmetric matrix that has an eigenvector basis.

3.2 Orthogonal matrices

In this subsection we look at some properties of orthogonal matrices. Remember that the columns of an orthogonal matrix form an orthonormal basis, not merely an orthogonal basis; that is, the columns are orthogonal vectors of magnitude 1.

We have said that whenever $\mathbf{P}$ is an orthogonal matrix we have $\mathbf{P}^T = \mathbf{P}^{-1}$. We now prove this result.

Theorem C65

A square matrix $\mathbf{P}$ is orthogonal if and only if $\mathbf{P}^T = \mathbf{P}^{-1}$.

Proof We know by Theorem C18 in Unit C1 that $\mathbf{P}^T\mathbf{P} = \mathbf{I}$ if and only if $\mathbf{P}\mathbf{P}^T = \mathbf{I}$, so $\mathbf{P}^T = \mathbf{P}^{-1}$ if and only if $\mathbf{P}^T\mathbf{P} = \mathbf{I}$.

So we need to show that $\mathbf{P}$ is orthogonal if and only if $\mathbf{P}^T\mathbf{P} = \mathbf{I}$. We start off by considering the expression $\mathbf{P}^T\mathbf{P}$.

Let the columns of the matrix $\mathbf{P}$ be the column vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. Then the rows of the matrix $\mathbf{P}^T$ are the row vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$.

For each i and j, the (i,j)-entry of $\mathbf{P}^T\mathbf{P}$ is the scalar product of the ith row of $\mathbf{P}^T$ and the jth column of $\mathbf{P}$; that is, $\mathbf{x}_i \cdot \mathbf{x}_j$.

So $\mathbf{P}^T\mathbf{P} = \mathbf{I}$ if and only if

$$\mathbf{x}_i \cdot \mathbf{x}_j = 0 \text{ whenever } i \neq j \quad \text{and} \quad \mathbf{x}_i \cdot \mathbf{x}_i = 1 \text{ for each } i.$$

This is the case precisely when $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is an orthonormal basis for $\mathbb{R}^n$; that is, when $\mathbf{P}$ is orthogonal. ■

Several properties of orthogonal matrices follow from Theorem C65.

Corollary C66

Let $\mathbf{P}$ and $\mathbf{Q}$ be orthogonal $n \times n$ matrices. Then:

(a) $\mathbf{P}^{-1}(=\mathbf{P}^T)$ is orthogonal

(b) the rows of $\mathbf{P}$ form an orthonormal basis for $\mathbb{R}^n$

(c) $\det \mathbf{P} = \pm 1$

(d) the product $\mathbf{PQ}$ is orthogonal.

Proof (a) To show that $\mathbf{P}^{-1}$ is orthogonal we must show that the transpose of $\mathbf{P}^{-1}$ is the inverse of $\mathbf{P}^{-1}$.

By Theorem C65 we have $\mathbf{P}^T = \mathbf{P}^{-1}$. Now,

$$(\mathbf{P}^{-1})^T\mathbf{P}^{-1} = (\mathbf{P}^T)^T\mathbf{P}^{-1} = \mathbf{PP}^{-1} = \mathbf{I}.$$

Thus $(\mathbf{P}^{-1})^T = (\mathbf{P}^{-1})^{-1}$, so $\mathbf{P}^{-1}(=\mathbf{P}^T)$ is orthogonal.

(b) The rows of $\mathbf{P}$ are the columns of $\mathbf{P}^T$. The matrix $\mathbf{P}^T$ is orthogonal by part (a), so its columns form an orthonormal basis for $\mathbb{R}^n$. Thus the rows of $\mathbf{P}$ form an orthonormal basis for $\mathbb{R}^n$.

(c) We know that $\det \mathbf{P}^T = \det \mathbf{P}$, and $\mathbf{P}^T = \mathbf{P}^{-1}$ by Theorem C65, so $\mathbf{P}^T\mathbf{P} = \mathbf{I}$.

By Theorem C14 in Unit C1 we know that $\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$ and $\det \mathbf{A}^T = \det \mathbf{A}$ for square matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size.

Now,

$$\det(\mathbf{P}^T\mathbf{P}) = (\det \mathbf{P}^T)(\det \mathbf{P}) = (\det \mathbf{P})^2,$$

but

$$\det(\mathbf{P}^T\mathbf{P}) = \det \mathbf{I} = 1,$$

so $(\det \mathbf{P})^2 = 1$. Hence $\det \mathbf{P} = \pm 1$.

(d) The proof of this is left for you to do in Exercise C139. ■

Exercise C139

Let $\mathbf{P}$ and $\mathbf{Q}$ be orthogonal $n \times n$ matrices. Prove that the product $\mathbf{PQ}$ is orthogonal.

(This is part (d) of Corollary C66.)

To understand why orthogonal diagonalisation is useful – beyond the ease of finding the inverse of the transition matrix – we will now look at the geometry of orthogonal transition matrices in $\mathbb{R}^2$ and $\mathbb{R}^3$.

We begin by asking to what transformations of the plane the 2×2 orthogonal matrices correspond. Suppose that

$$\mathbf{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an orthogonal matrix. Then the vectors (a, c) and (b, d) form an orthonormal basis for $\mathbb{R}^2$ and $\det \mathbf{P} = \pm 1$.

We stated in Subsection 3.2 of Unit C3 that the magnitude of the determinant of a matrix of a linear transformation gives the 'scaling factor'. Therefore $\det \mathbf{P} = \pm 1$ means that there is no scaling; that is, magnitudes are preserved.

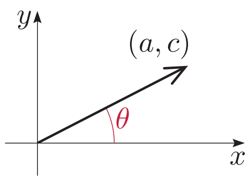


Figure 10 The angle θ made by the vector (a, c)

Let θ be the angle that the unit vector (a, c) makes with the x-axis, as illustrated in Figure 10 for the case that (a, c) is in the first quadrant, so

$$(a, c) = (\cos\theta, \sin\theta).$$

Since the unit vector (b, d) is orthogonal to (a, c), we have $(a, c) \cdot (b, d) = 0$, so

$$(b, d) = (-\sin\theta, \cos\theta) \quad \text{or} \quad (\sin\theta, -\cos\theta),$$

as illustrated in Figure 11.

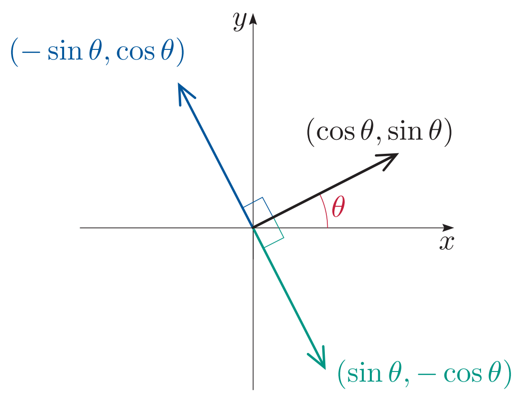


Figure 11 The two possible vectors (b, d) orthogonal to the vector (a, c)

Hence, if $\det \mathbf{P} = +1$, then

$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

and if $\det \mathbf{P} = -1$, then

$$\mathbf{P} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

Now suppose that $E = \{\mathbf{e}_1, \mathbf{e}_2\}$ is an orthonormal basis for $\mathbb{R}^2$ and that $\mathbf{P}$ is the orthogonal transition matrix whose columns are formed from the coordinates of $\mathbf{e}_1$ and $\mathbf{e}_2$.

We have just seen that if $\det \mathbf{P} = +1$, then

$$\mathbf{e}_1 = (\cos\theta, \sin\theta) \quad \text{and} \quad \mathbf{e}_2 = (-\sin\theta, \cos\theta),$$

that is, $\mathbf{e}_1$ and $\mathbf{e}_2$ are the images of the standard basis vectors $(1, 0)$ and $(0, 1)$ under a rotation r_θ, as illustrated in Figure 12.

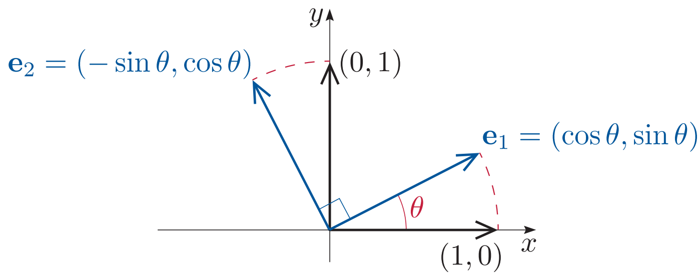


Figure 12 A rotation r_θ

Similarly, if $\det \mathbf{P} = -1$, then $\mathbf{e}_1$ and $\mathbf{e}_2$ are the images of the standard basis vectors $(1, 0)$ and $(0, 1)$ under a reflection $q_{\theta/2}$, as illustrated in Figure 13.

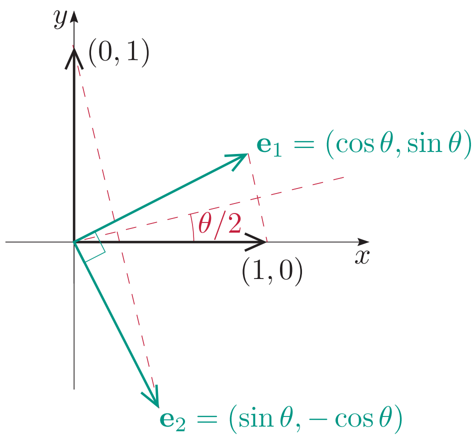


Figure 13 A reflection $q_{\theta/2}$

So if a 2×2 orthogonal matrix $\mathbf{P}$ is used to represent a linear transformation (as opposed to a transition matrix), then the linear transformation must be either a rotation or a reflection.

Similar arguments can be applied to 3×3 orthogonal matrices to show that linear transformations of $\mathbb{R}^3$ whose matrices are orthogonal are rotations about a line through the origin, reflections in a plane through the origin or combinations of these. The orthogonal matrices representing rotations of $\mathbb{R}^3$ are precisely those with determinant $+1$.

Exercise C140

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(a) Verify that this matrix is orthogonal.

(b) Show that this matrix represents a rotation of $\mathbb{R}^3$.

Let t be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ with a matrix representation that is a symmetric matrix $\mathbf{A}$. In effect, when we orthogonally diagonalise $\mathbf{A}$, we are finding a basis for $\mathbb{R}^n$ for which

- the matrix of t is diagonal
- the basis vectors are orthogonal
- the basis vectors have magnitude 1.

For $\mathbb{R}^2$ and $\mathbb{R}^3$ this new basis is simply the standard basis rotated, reflected or, for $\mathbb{R}^3$, a combination of the two.

4 Conics and quadrics

In this section you will classify conics and quadrics using many of the techniques you have learned in this book on linear algebra, including orthogonal diagonalisation of symmetric matrices.

You revised conics in Unit A4 *Real functions, graphs and conics*.

4.1 Classifying conics

A non-degenerate conic may be a circle, an ellipse, a parabola or a hyperbola. It is said to be in **standard position** if it is positioned in the plane as follows.

- For a circle: its centre is at the origin.
- For an ellipse: its axes of symmetry are the x- and y-axes, and its largest width is along the x-axis.
- For a parabola: its axis of symmetry is the x-axis, it passes through the origin and its other points lie to the right of the origin.
- For a hyperbola: its axes of symmetry are the x- and y-axes, and it crosses the x-axis.

A circle may sometimes be considered to be a special type of ellipse, and that will be the case throughout this section.

An ellipse, a parabola and a hyperbola in standard position are illustrated in Figure 14.

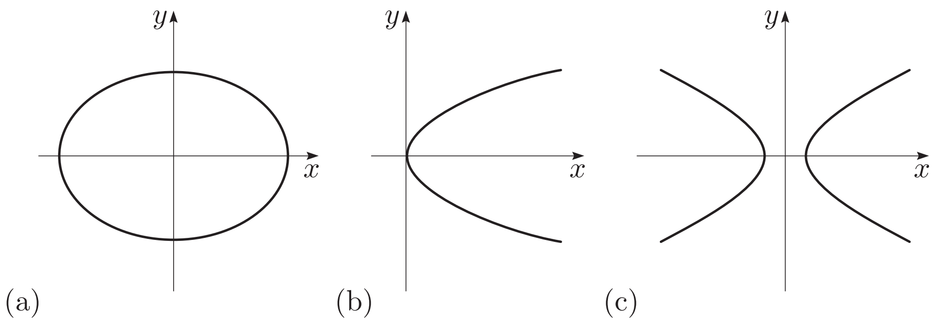


Figure 14 Conics in standard position: (a) ellipse (b) parabola and (c) hyperbola

The line joining the vertices of an ellipse is the major axis of the ellipse, and the line perpendicular to this through the centre of the ellipse is the minor axis of the ellipse. Thus, for an ellipse in standard position, the major and minor axes are the x-axis and y-axis, respectively.

We can define major and minor axes for parabolas and hyperbolas similarly.

- For a parabola, the major axis is the axis of the parabola, and the minor axis is the line perpendicular to this through the vertex of the parabola.
- For a hyperbola, the major axis is the line joining the vertices of the hyperbola, and the minor axis is the line perpendicular to this through the centre of the hyperbola.

Notice, in each case the minor axis is parallel to the directrix of the conic. (You met the directrix of a conic in Section 5 of Unit A4).

In this way, the major and minor axes of any conic in standard position are the x-axis and y-axis, respectively.

An ellipse in standard position has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

a parabola in standard position has equation

$$y^2 = 4ax$$

and a hyperbola in standard position has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Theorem A21 in Unit A4 says that any conic in $\mathbb{R}^2$ is the set of points (x, y) in $\mathbb{R}^2$ that satisfy an equation of the following form

$$Ax^2 + Bxy + Cy^2 + Fx + Gy + H = 0, \qquad (6)$$

where A, B, C, F, G and H are real numbers, and A, B and C are not all zero. This theorem also says the converse: that the set of all points in $\mathbb{R}^2$ whose coordinates (x, y) satisfy an equation of this form is a conic. However, such a conic may be degenerate – in this subsection we will only be concerned with non-degenerate conics.

Given the equation of a non-degenerate conic, such as

$$x^2 - 4xy - 2y^2 + 6x + 12y + 21 = 0, \qquad (7)$$

we would like to be able to decide whether it represents an ellipse, a hyperbola or a parabola. We know it is not a circle because of the non-zero term in xy, but it is too complicated to easily determine more than this. Generally, the equations of conics that arise in calculations are not in standard position: thus we need some way of determining the nature of a conic from its equation.

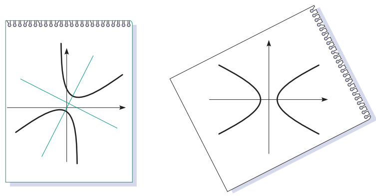


Figure 15 Moving the axes to be able to recognise a conic

In fact, equation (7) represents a hyperbola with centre $(1, 2)$, major axis $y = 2x$ and minor axis $x + 2y = 5$. This conic would be easily recognisable were we to *move the axes of the plane* so that they pass through the centre and line up with the major and minor axes of the conic, as illustrated in Figure 15.

You will see that we can move the axes of the plane by introducing matrices and changing the basis for the plane, then performing a translation so that the conic is in standard position with respect to these new basis vectors. The conic will then be easily recognisable from its equation.

We will actually be a little less specific with how we move the axes mathematically and may not always end up with a conic in standard position: the axes may be interchanged or pointing in the opposite directions resulting in conics that are reflected or rotated. However, in every case the axes will align with the major and minor axes of the conic, and the equation will resemble the equation of a conic in standard position; we say that such an equation of a conic is in **standard form**.

An ellipse and a hyperbola with equations in standard form, but that are not in standard position, are illustrated in Figure 16.

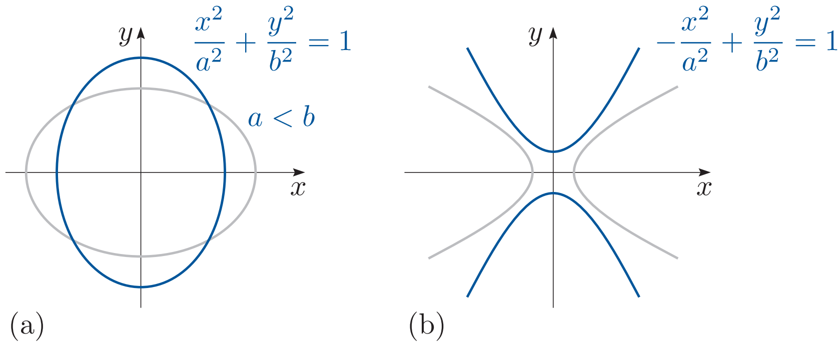


Figure 16 Conics not in standard position with equations in standard form: (a) ellipse and (b) hyperbola

Parabolas with equations in standard form, but not in standard position, are illustrated in Figure 17.

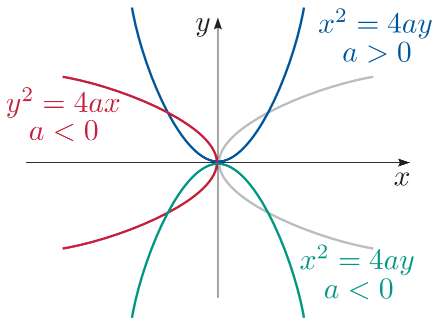


Figure 17 Parabolas not in standard position with equations in standard form

Introducing matrices

We first write equation (6) $Ax^2+Bxy+Cy^2+Fx+Gy+H=0$ using matrices and vectors; that is, in matrix form as

$$\mathbf{x}^T\mathbf{A}\mathbf{x}+\mathbf{J}^T\mathbf{x}+H=0, \tag{8}$$

where

$$\mathbf{A}=\begin{pmatrix} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{pmatrix}, \quad \mathbf{J}=\begin{pmatrix} F \\ G \end{pmatrix}, \quad \mathbf{x}=\begin{pmatrix} x \\ y \end{pmatrix}.$$

This is possible, since

$$\begin{aligned}\mathbf{x}^T\mathbf{A}\mathbf{x}=\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} Ax+\frac{1}{2}By \\ \frac{1}{2}Bx+Cy \end{pmatrix} \\ &= Ax^2+Bxy+Cy^2\end{aligned}$$

and

$$\begin{aligned}\mathbf{J}^T\mathbf{x} &= \begin{pmatrix} F & G \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= Fx+Gy.\end{aligned}$$

Notice that the matrix $\mathbf{A}$ is symmetric; this will be important.

For example, the conic with equation (7) can be written in matrix form (8) with

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -2 & -2 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad H = 21.$$

Exercise C141

For each of the following equations of a conic in standard position, write the equation in matrix form and specify the matrices $\mathbf{A}$ and $\mathbf{J}$.

(a) the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (b) the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

(c) the parabola $y^2 = 4ax$

Aligning the axes

The matrix $\mathbf{A}$ in the matrix representation (8) is symmetric, so we know that we can orthogonally diagonalise this matrix to get $\mathbf{P}^T\mathbf{AP} = \mathbf{D}$ where $\mathbf{P}$ is an orthogonal transition matrix.

This helps us recognise the conic by aligning the basis vectors with the axes of the conic and therefore removing the xy-terms from the equation. The columns of $\mathbf{P}$ form an orthonormal basis E, and $\mathbf{P}$ changes E-coordinates $\mathbf{x}_E$, which we will write in the form $\mathbf{x}' = (x', y')$, into standard coordinates $\mathbf{x} = (x, y)$, so that $\mathbf{x} = \mathbf{P}\mathbf{x}'$.

In this way equation (8) becomes

$$(\mathbf{P}\mathbf{x}')^T\mathbf{A}(\mathbf{P}\mathbf{x}') + \mathbf{J}^T\mathbf{P}\mathbf{x}' + H = 0,$$

which can be rewritten as

$$(\mathbf{x}')^T(\mathbf{P}^T\mathbf{AP})\mathbf{x}' + \mathbf{J}^T\mathbf{P}\mathbf{x}' + H = 0. \tag{9}$$

Now, $\mathbf{P}^T\mathbf{AP} = \mathbf{D}$ is a diagonal matrix with diagonal entries λ_1 and λ_2, so we have

$$\begin{aligned} (\mathbf{x}')^T(\mathbf{P}^T\mathbf{AP})\mathbf{x}' &= (\mathbf{x}')^T\mathbf{D}\mathbf{x}' \\ &= \begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \\ &= \lambda_1(x')^2 + \lambda_2(y')^2, \end{aligned}$$

and therefore there is no $x'y'$-term in the new equation (9) for the conic. Written in the form of equation (9), this now more closely resembles the equation of a conic in standard position. The vectors in the orthonormal basis E of the plane are aligned with the axes of the conic: we say we have *aligned the axes*.

The order and direction in which the eigenvectors are chosen affects the orthonormal basis E and therefore the transition matrix $\mathbf{P}$ obtained.

However, in every case $\mathbf{P}$ is an orthogonal matrix and so $\det \mathbf{P} = \pm 1$. Orthogonal diagonalisation ensures that the new basis vectors are orthogonal (perpendicular) and of magnitude 1. If $\mathbf{P}$ is considered to represent a linear transformation (as opposed to a transition matrix), then the linear transformation is either a rotation ($\det \mathbf{P} = +1$) or a reflection ($\det \mathbf{P} = -1$).

It is sometimes preferable, when choosing the orthonormal basis E, for it to be a rotation (rather than a reflection) of the standard basis vectors; that is, that $\mathbf{P}$, considered as a linear transformation, is a rotation. This is achieved by ensuring that $\det \mathbf{P} = +1$ (using either geometric insight, or by checking the determinant). However, this step is not required in this module.

We now illustrate the process of rewriting a conic in the form of equation (9) by applying the process to equation (7), where

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -2 & -2 \end{pmatrix}.$$

Worked Exercise C73

Express the non-degenerate conic

$$x^2 - 4xy - 2y^2 + 6x + 12y + 21 = 0$$

in the form of equation (9).

Solution

The matrix form of the equation of the conic is $\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{J}^T\mathbf{x} + H = 0$ where

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -2 & -2 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad H = 21.$$

In Exercise C119(b) you found that the eigenvectors of $\mathbf{A}$ are the non-zero vectors $(k, 2k)$ and $(-2k, k)$, corresponding to the eigenvalues $\lambda = -3$ and $\lambda = 2$, respectively.

We start by orthogonally diagonalising $\mathbf{A}$.

We use Strategy C22 to orthogonally diagonalise $\mathbf{A}$.

An orthonormal basis for $S(-3)$ is

$$\left\{\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)\right\},$$

and an orthonormal basis for $S(2)$ is

$$\left\{\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)\right\}.$$

By Theorem C64 an orthonormal eigenvector basis of $\mathbf{A}$ is therefore

$$E = \left\{ \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right), \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right) \right\}.$$

We use the eigenvectors in E to form the columns of the transition matrix:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}.$$

Note that $\det \mathbf{P} = +1$, so the basis vectors in E are the images of the standard basis vectors under a rotation, but that does not concern us here.

We use the eigenvalues to form the diagonal matrix

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}.$$

We substitute into $(\mathbf{x}')^T(\mathbf{P}^T\mathbf{A}\mathbf{P})\mathbf{x}' + \mathbf{J}^T\mathbf{P}\mathbf{x}' + H = 0$.

It follows from equation (9) that the equation of the conic is now

$$\begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} 6 & 12 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + 21 = 0,$$

that is,

$$-3(x')^2 + 2(y')^2 + 6\sqrt{5}x' + 21 = 0.$$

There are no terms in $x'y'$ in this new equation.

You might wonder what the equation in Worked Exercise C73 would have been if the eigenvalues had been chosen in the opposite order? The next exercise investigates this.

Exercise C142

Express the non-degenerate conic

$$x^2 - 4xy - 2y^2 + 6x + 12y + 21 = 0$$

in the form of equation (9), using the eigenvalues in the order $\lambda = 2$ then $\lambda = -3$.

The equation of the conic with the eigenvalues $\lambda = -3$ then $\lambda = 2$ and the equation of the conic with the eigenvalues $\lambda = 2$ then $\lambda = -3$ are very similar. It looks like the roles of x' and y' have been interchanged; that is, the order of the coordinates have been interchanged, which corresponds to interchanging the axes. We have $\det \mathbf{P} = -1$ in Exercise C142 so this transition matrix corresponds to a reflection of the axes, whereas we have $\det \mathbf{P} = 1$ in Worked Exercise C73 so this transition matrix corresponds to a rotation.

In general, for any conic, if

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

then equation (9) is of the form

$$\lambda_1(x')^2 + \lambda_2(y')^2 + fx' + gy' + H = 0, \tag{10}$$

where $\begin{pmatrix} f & g \end{pmatrix} = \mathbf{J}^T\mathbf{P}$.

The equation of the conic in this form has been simplified since it now has no $x'y'$ terms, but is not yet in a form from which we can easily recognise the type of the conic: a translation of the axes is also required.

Translating the origin

To write the equation of the conic in standard form from which we can easily recognise the type of the conic, we need to eliminate any superfluous linear x' and y' terms. This is achieved by translating the origin using an (α, β)-translation and moving to new coordinates $\mathbf{x}'' = (x'', y'')$: we say we have *translated the origin*.

To do this, we first *complete the squares* in the equation of the conic. We illustrate this process using the conic with equation (7). We have already aligned the axes to obtain the equation

$$-3(x')^2 + 2(y')^2 + 6\sqrt{5}x' + 21 = 0,$$

which is equivalent to

$$-3\left((x')^2 - 2\sqrt{5}x'\right) + 2(y')^2 + 21 = 0.$$

This equation has no linear y' term, so we only need to complete the square involving x'. We obtain

$$-3(x' - \sqrt{5})^2 + 15 + 2(y')^2 + 21 = 0,$$

so

$$-3(x' - \sqrt{5})^2 + 2(y')^2 + 36 = 0.$$

In Subsection 1.3 of Unit A4 you saw that applying an (α, β)-translation to the graph of $y = f(x)$ gives the graph of $y = f(x - \alpha) + \beta$, or equivalently, $y - \beta = f(x - \alpha)$. We can express this translated curve more simply by using new (x', y')-coordinates obtained by an (α, β)-translation of the (x, y)-axes: we do this by setting $x' = x - \alpha$ and $y' = y - \beta$. In this new (x', y')-coordinate system the equation of the translated curve is $y' = f(x')$.

For our conic we use a $(\sqrt{5},0)$-translation, so we set the new coordinates to be

$$\mathbf{x}'' = (x'', y'') = (x' - \sqrt{5}, y').$$

Thus we rewrite the equation of the conic using these coordinates by substituting

$$x'' = x' - \sqrt{5} \quad \text{and} \quad y'' = y',$$

which results in the following simplified equation of the conic

$$-3(x'')^2 + 2(y'')^2 = -36,$$

or

$$\frac{(x'')^2}{12} - \frac{(y'')^2}{18} = 1.$$

This equation is now recognisable as the equation of a hyperbola in standard form. In fact, it is also a hyperbola in standard position with respect to these new axes, since the $(x'')^2$ term is positive and the $(y'')^2$ term is negative.

For this conic we have

- introduced matrices $\mathbf{A}$ and $\mathbf{J}$
- orthogonally diagonalised the matrix $\mathbf{A}$ to find the orthogonal transition matrix $\mathbf{P}$ which rotates the (x,y)-axes by $\theta = \cos^{-1}(1/\sqrt{5})$ to get the (x',y')-axes
- translated by $\sqrt{5}$ in the x' direction to get the (x'',y'')-axes.

This is illustrated in Figure 18.

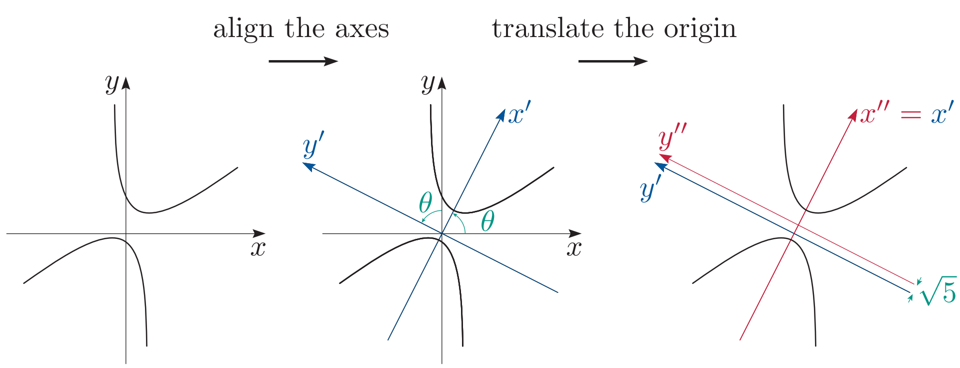


Figure 18 Moving the axes to get the equation of the conic in standard form ($\lambda = -3$ then $\lambda = 2$)

What would the equation of this conic have been if the eigenvalues had been chosen in the opposite order? The next exercise investigates this using the equation you found in Exercise C142.

Exercise C143

Write the equation of the conic

$$x^2 - 4xy - 2y^2 + 6x + 12y + 21 = 0$$

in standard form by completing the square in the equation

$$2(x')^2 - 3(y')^2 + 6\sqrt{5}y' + 21 = 0$$

and then making a substitution to get coordinates (x'', y'').

Figure 19 illustrates how the axes have been moved with the eigenvalues in the order $\lambda = 2$ then $\lambda = -3$, as in Exercise C143: the axes are reflected and then translated.

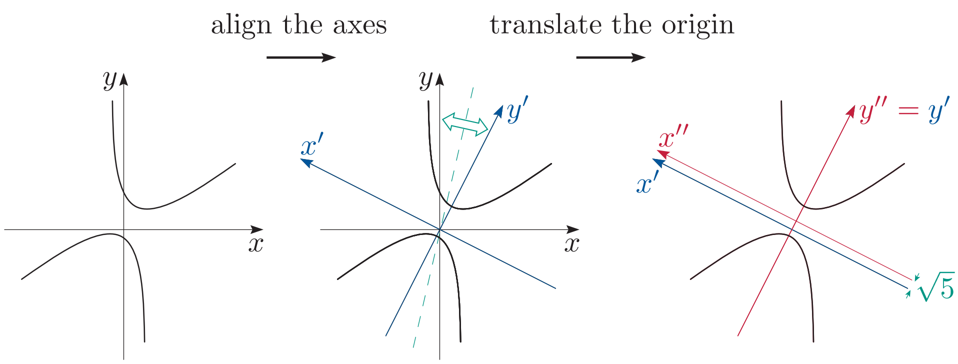


Figure 19 Moving the axes to get the equation of the conic in standard form ($\lambda = 2$ then $\lambda = -3$)

The equations in standard form found for the conic with equation (7) are

$$\frac{(x'')^2}{12} - \frac{(y'')^2}{18} = 1, \quad \text{for } \lambda = -3 \text{ then } \lambda = 2,$$

and

$$-\frac{(x'')^2}{18} + \frac{(y'')^2}{12} = 1, \quad \text{for } \lambda = 2 \text{ then } \lambda = -3.$$

In the second case the hyperbola is not in standard position with respect to these new axes, since the $(x'')^2$ term is negative and the $(y'')^2$ term is positive.

It is clear that the roles of x'' and y'' have been interchanged. Geometrically, the new axes of the plane have been interchanged, so the hyperbola has related, but different, equations in relation to these different choices of axes. However, both equations are in the standard form for a hyperbola, so the choice of the order of the eigenvalues does not affect the conclusion that this conic is a hyperbola.

Ellipse and hyperbola

In general, if neither eigenvalue is 0, then completing the squares in equation (10) gives an equation of the form

$$\lambda_1\left(x'+\frac{f}{2\lambda_1}\right)^2-\lambda_1\left(\frac{f}{2\lambda_1}\right)^2+\lambda_2\left(y'+\frac{g}{2\lambda_2}\right)^2-\lambda_2\left(\frac{g}{2\lambda_2}\right)^2+H=0,$$

which can be written as

$$\lambda_1(x'')^2+\lambda_2(y'')^2=K,$$

where

$$x''=x'+\frac{f}{2\lambda_1},\quad y''=y'+\frac{g}{2\lambda_2}\quad\text{and}\quad K=\frac{f^2}{4\lambda_1}+\frac{g^2}{4\lambda_2}-H.$$

Writing the equation in standard form gives

$$\frac{(x'')^2}{K/\lambda_1}+\frac{(y'')^2}{K/\lambda_2}=1,$$

which is the equation of an ellipse if both K/λ_1 and K/λ_2 are positive, and a hyperbola if one is negative and the other positive. (No other possibility can occur, although we do not explicitly show this.)

Parabola

In general, if one eigenvalue is 0, say λ_1 is 0 and $\lambda_2 \neq 0$, then equation (10) has the form

$$\lambda_2(y')^2+fx'+gy'+H=0.$$

Completing the square in this equation gives

$$fx'+\lambda_2\left(y'+\frac{g}{2\lambda_2}\right)^2-\lambda_2\left(\frac{g}{2\lambda_2}\right)^2+H=0,$$

which can be written as

$$\lambda_2(y'')^2+fx''=0,$$

where

$$y''=y'+\frac{g}{2\lambda_2},\quad x''=x'-\frac{\lambda_2}{f}\left(\frac{g}{2\lambda_2}\right)^2+\frac{H}{f}.$$

Writing the equation in standard form gives

$$(y'')^2=\frac{-f}{\lambda_2}x'',$$

which is the equation of a parabola.

If $\lambda_1 \neq 0$ and λ_2 is 0, then we obtain the similar equation

$$(x'')^2=\frac{-g}{\lambda_1}y'',$$

which is also the equation of a parabola.

Summarising the method

There are several steps involved in writing the equation of a conic in standard form, so we summarise this method in the following strategy.

Strategy C24

To write the non-degenerate conic with equation

$$Ax^2 + Bxy + Cy^2 + Fx + Gy + H = 0$$

in standard form, do the following.

1. Introduce matrices:
   - write down $\mathbf{A} = \begin{pmatrix} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{pmatrix}$ and $\mathbf{J} = \begin{pmatrix} F \\ G \end{pmatrix}$.
2. Align the axes:
   - orthogonally diagonalise $\mathbf{A}$ to get
     $$\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$
   - find $\begin{pmatrix} f & g \end{pmatrix} = \mathbf{J}^T\mathbf{P}$, and write the conic in the form
     $$\lambda_1(x')^2 + \lambda_2(y')^2 + fx' + gy' + H = 0.$$
3. Translate the origin:
   - complete the squares
   - make a substitution to change to the coordinate system (x'', y'').

The order in which the eigenvalues are chosen does not affect the *form* of the equation obtained: it will be the standard form for an ellipse, a hyperbola or a parabola.

The following worked exercise and exercises illustrate this strategy.

Worked Exercise C74

Use Strategy C24 to write the non-degenerate conic with equation

$$5x^2 + 4xy + 5y^2 + 20x + 8y - 1 = 0$$

in standard form. Is this conic an ellipse, a parabola or a hyperbola?

Solution

Since some parts of this working can be quite long, we number the strategy steps in the solution.

1. Introduce matrices.
   We have
   $$\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} 20 \\ 8 \end{pmatrix}.$$

2. Align the axes.

We orthogonally diagonalised $\mathbf{A}$ in Worked Exercise C71.

We have

$$\mathbf{P}^T\mathbf{AP} = \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix},$$

where

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix},$$

so

$$\begin{aligned}
\begin{pmatrix} f & g \end{pmatrix} &= \begin{pmatrix} 20 & 8 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{28}{\sqrt{2}} & \frac{12}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} 14\sqrt{2} & 6\sqrt{2} \end{pmatrix}.
\end{aligned}$$

The equation of the conic is now

$$7(x')^2 + 3(y')^2 + 14\sqrt{2}x' + 6\sqrt{2}y' - 1 = 0.$$

3. Translate the origin.

To keep track of the terms when completing the square, we first collect the x' terms and the y' terms. We take out the coefficients of $(x')^2$ and $(y')^2$ as factors.

We write this equation as

$$7\left((x')^2 + 2\sqrt{2}x'\right) + 3\left((y')^2 + 2\sqrt{2}y'\right) - 1 = 0.$$

Completing the squares in this equation, we obtain

$$7(x' + \sqrt{2})^2 - 14 + 3(y' + \sqrt{2})^2 - 6 - 1 = 0.$$

We substitute $x'' = x' + \sqrt{2}$ and $y'' = y' + \sqrt{2}$ into this equation and simplify to obtain

$$7(x'')^2 + 3(y'')^2 - 21 = 0.$$

The equation of the conic in standard form is

$$\frac{(x'')^2}{3} + \frac{(y'')^2}{7} = 1.$$

The conic is an ellipse.

We can see that this ellipse is not in standard position with respect to these new axes since $3 < 7$.

Exercise C144

Use Strategy C24 to write the non-degenerate conic with equation

$$9x^2 - 4xy + 6y^2 - 10x - 20y - 5 = 0$$

in standard form. Is the conic an ellipse, a parabola or a hyperbola?

(In Exercise C137(a) you found that

$$E = \left\{ \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right), \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) \right\}$$

is an orthonormal eigenvector basis for the matrix $\mathbf{A}$ of this conic with respect to the eigenvalues $\lambda = 10$ and $\lambda = 5$.)

Exercise C145

Use Strategy C24 to write the non-degenerate conic with equation

$$x^2 - 4xy + 4y^2 - 6x - 8y + 5 = 0$$

in standard form. Is the conic an ellipse, a parabola or a hyperbola?

4.2 Classifying quadrics

Quadrics, or *quadric surfaces*, are surfaces in $\mathbb{R}^3$. They are the three-dimensional analogues of conics.

Definition

A **quadric** in $\mathbb{R}^3$ is the set of points (x, y, z) that satisfy an equation of the form

$$Ax^2 + By^2 + Cz^2 + Fxy + Gyz + Hxz + Jx + Ky + Lz + M = 0,$$

where A to M are real numbers, and A, B, C, F, G and H are not all 0.

In general the situation is more complicated than for conics and the general situation is beyond the scope of this module. However, it can be shown that there are *nine* types of quadrics involving curved surfaces in $\mathbb{R}^3$. Each of these types can be positioned in space to be in **standard position**; that is, with its axes aligned with the x-, y- and z-axes in a similar manner to the non-degenerate conics. These quadrics in standard position have easily recognisable equations and the different types can be distinguished by the **curves of intersection** of the planes parallel to the coordinate planes that meet the quadric in a non-trivial intersection. Figure 20 shows some curves of intersection for a sphere – they are all circles.

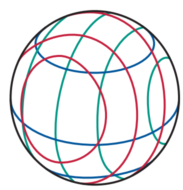


Figure 20 Some curves of intersection of a sphere

The curves of intersection of a **non-degenerate quadric** are non-degenerate conics. There are five types of non-degenerate quadric:

- the *ellipsoid* (which includes the sphere)
- the *elliptic paraboloid*
- the *hyperbolic paraboloid*
- the *hyperboloid of one sheet*
- the *hyperboloid of two sheets*.

Table 1 illustrates each of these quadrics and gives the equation in standard position, as well as specifying the curves of intersection.

There are four types of **degenerate quadric** involving curved surfaces:

- the *elliptic cone*
- the *elliptic cylinder*
- the *parabolic cylinder*
- the *hyperbolic cylinder*.

The curves of intersection of these include non-degenerate conics, degenerate conics and pairs of parallel lines. The elliptic cone in standard position is illustrated in Table 1, where the equation is given and the curves of intersection specified. The elliptic cone can be considered as intermediate between the hyperboloids of one and two sheets – where the two sheets touch at a point. The three types of cylinder in standard position, illustrated in Figure 21, are surfaces whose equations do not involve z explicitly.

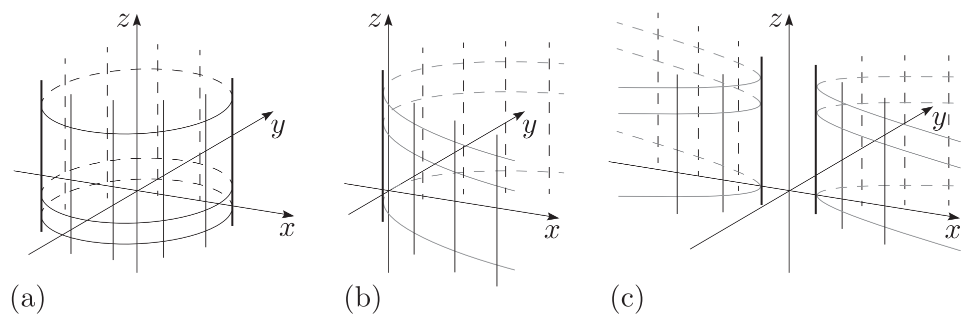
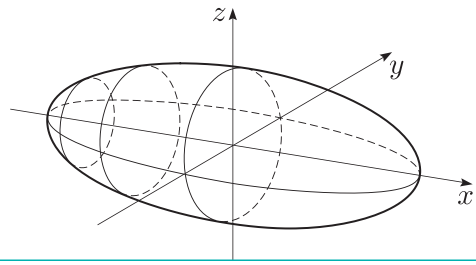
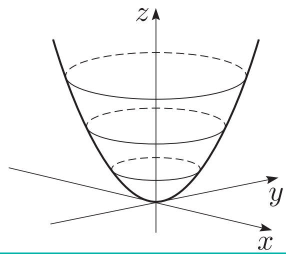
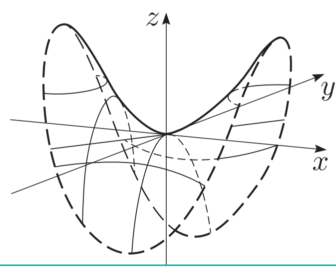
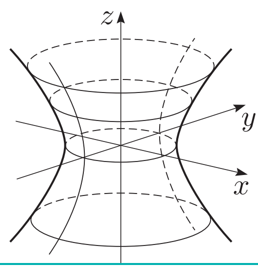
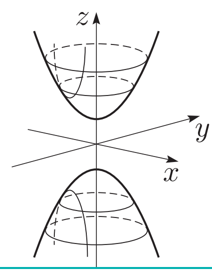
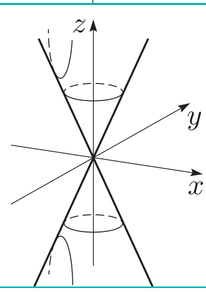


Figure 21 Degenerate quadrics: (a) elliptic cylinder (b) parabolic cylinder and (c) hyperbolic cylinder

The only degenerate quadrics we will consider for the remainder of the linear algebra topic are elliptic cones, thus giving the following list of *six quadrics*, all included in Table 1: the ellipsoid (including the sphere), the elliptic paraboloid, the hyperbolic paraboloid, the hyperboloid of one sheet, the hyperboloid of two sheets and the elliptic cone.

Table 1 Quadrics: equation in standard position and the curves of intersection

Ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ curves of intersection: ellipse	
Elliptic paraboloid $z=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$ curves of intersection: ellipse or parabola	
Hyperbolic paraboloid $z=\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}$ curves of intersection: hyperbola or parabola	
Hyperboloid of one sheet $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$ curves of intersection: ellipse or hyperbola	
Hyperboloid of two sheets $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=-1$ curves of intersection: ellipse or hyperbola	
Elliptic cone $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=0$ curves of intersection: ellipse or hyperbola (or a degenerate conic)	

Gaspard Monge

Jean Nicolas Pierre Hachette

The first systematic classification of quadric surfaces was by Leonhard Euler (1707–1783) in his celebrated *Introductio in analysin infinitorum* (1748) – the textbook in which he laid down the foundations of analysis – where he treated surfaces of second degree as a family of quadrics in space analogous to the plane conic sections. The subject was developed in a more rigorous way by Gaspard Monge (1746–1818) and Jean Nicolas Pierre Hachette (1769–1834) who, in 1802, provided an algebraic study of quadric surfaces, which was later published as a textbook. Both Monge and Hachette were professors at the famous École Polytechnique in Paris. This college was founded at the end of the nineteenth century to provide students with a mathematical and scientific education, and to prepare them for entry to the prestigious Grandes Écoles, higher education establishments for the training of civil and military engineers.

As with conics, to identify a given quadric from its equation, we will align the axes and translate the origin to obtain an equation that resembles the equation of a quadric in standard position: we say that such an equation of a quadric is in **standard form**. So, for example, the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is an equation of a hyperboloid of one sheet in standard form, although it is not in standard position.

To write the equation of a quadric in standard form, we use the same techniques that we used for conics: introducing matrices, orthogonal diagonalisation and completing the square. We omit the justification – it is analogous to that for conics.

We summarise this method in the following strategy.

Strategy C25

To write the quadric with equation

$$Ax^2 + By^2 + Cz^2 + Fxy + Gyz + Hxz + Jx + Ky + Lz + M = 0$$

in standard form, do the following.

1. Introduce matrices:
   - write down the matrices

$$\mathbf{A} = \begin{pmatrix} A & \frac{1}{2}F & \frac{1}{2}H \\ \frac{1}{2}F & B & \frac{1}{2}G \\ \frac{1}{2}H & \frac{1}{2}G & C \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} J \\ K \\ L \end{pmatrix}.$$

2. Align the axes:
   - orthogonally diagonalise $\mathbf{A}$ to get
   $$\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$
   - find $\begin{pmatrix} f & g & h \end{pmatrix} = \mathbf{J}^T\mathbf{P}$, and write the quadric in the form $\lambda_1(x')^2 + \lambda_2(y')^2 + \lambda_3(z')^2 + fx' + gy' + hz' + M = 0$.
3. Translate the origin:
   - complete the squares
   - make a substitution to change to the coordinate system (x'', y'', z'').

The following worked exercise and exercises illustrate this strategy.

Worked Exercise C75

Use Strategy C25 to write the quadric with equation
$$5x^2 + 3y^2 + 3z^2 - 2xy + 2yz - 2xz - 10x + 6y - 2z - 9 = 0$$
in standard form. Which of the six types of quadric does this represent?

Solution

As with conics, since some parts of this working can be quite long, we number the strategy steps in the solution.

1. Introduce matrices.

   We have
   $$\mathbf{A} = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} -10 \\ 6 \\ -2 \end{pmatrix}.$$
2. Align the axes.

   You orthogonally diagonalised $\mathbf{A}$ in Exercise C137(b).

   We have
   $$\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$
   where
   $$\mathbf{P} = \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Since $\det \mathbf{P} = 1$, this transition matrix represents a rotation of the basis vectors, but this fact does not concern us here.

So

$$\begin{aligned}\begin{pmatrix} f & g & h \end{pmatrix} &= \begin{pmatrix} -10 & 6 & -2 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} 4\sqrt{6} & -2\sqrt{3} & 4\sqrt{2} \end{pmatrix}.\end{aligned}$$

The equation of the quadric is now

$$6(x')^2 + 3(y')^2 + 2(z')^2 + 4\sqrt{6}x' - 2\sqrt{3}y' + 4\sqrt{2}z' - 9 = 0.$$

3. Translate the origin.

We write this equation as

$$6\left((x')^2 + \frac{4}{\sqrt{6}}x'\right) + 3\left((y')^2 - \frac{2}{\sqrt{3}}y'\right) + 2\left((z')^2 + 2\sqrt{2}z'\right) - 9 = 0.$$

Completing the squares in this equation, we obtain

$$6\left(x' + \frac{2}{\sqrt{6}}\right)^2 - 4 + 3\left(y' - \frac{1}{\sqrt{3}}\right)^2 - 1 + 2(z' + \sqrt{2})^2 - 4 - 9 = 0.$$

Substituting

$$x'' = x' + \frac{2}{\sqrt{6}}, \quad y'' = y' - \frac{1}{\sqrt{3}} \quad \text{and} \quad z'' = z' + \sqrt{2}$$

in this equation and simplifying, we obtain

$$6(x'')^2 + 3(y'')^2 + 2(z'')^2 - 18 = 0.$$

The equation of the quadric in standard form is

$$\frac{(x'')^2}{3} + \frac{(y'')^2}{6} + \frac{(z'')^2}{9} = 1.$$

This is the equation of an ellipsoid.

Exercise C146

Use Strategy C25 to write the quadric with equation

$$x^2 + y^2 + z^2 - 2x + 4y - 6z - 11 = 0$$

in standard form. Which of the six types of quadric does this represent?

Exercise C147

Use Strategy C25 to write the quadric with equation

$$4x^2 + 3y^2 + 2z^2 + 4xy + 4yz + 12x + 12z + 18 = 0$$

in standard form. Which of the six types of quadric does this represent?

(At the start of Subsection 3.1 we found that

$$E = \left\{ \left(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}\right), \left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}\right), \left(\tfrac{1}{3}, -\tfrac{2}{3}, \tfrac{2}{3}\right) \right\}$$

is an orthonormal eigenvector basis for the matrix $\mathbf{A}$ of this quadric with respect to the eigenvalues $\lambda = 6$, $\lambda = 3$ and $\lambda = 0$.)

Summary

In this unit you have met eigenvectors and eigenvalues: an eigenvector of a linear transformation $t : V \to V$ is a non-zero vector $\mathbf{v}$ that is mapped by t to a scalar multiple of itself, and this scalar is the corresponding eigenvalue λ. Since such a linear transformation always has a square matrix representation, you have seen that eigenvectors and eigenvalues can equivalently be defined in terms of matrices: $\mathbf{Av} = \lambda\mathbf{v}$. You have found eigenvalues and eigenvectors by solving the corresponding characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. You have seen that there may be no eigenvalues, for example when t is a rotation of $\mathbb{R}^2$, and that all the eigenvectors corresponding to a given eigenvalue λ, plus the zero vector, form a subspace $S(\lambda)$ of V whose dimension is never greater than the multiplicity of the eigenvalue.

You have investigated when t has an eigenvector basis E; that is, a basis comprising only eigenvectors of t, and you have met transition matrices $\mathbf{P}$ that map a basis E of V to the standard basis. You have seen (Theorem C60) that the transition matrix $\mathbf{P}$ maps standard coordinates of V to E-coordinates of V and that $\mathbf{P}$ is invertible. You have learned (Theorem C62) that whenever an eigenvector basis can be found, the transition matrix $\mathbf{P}$ can be used to express the matrix $\mathbf{A}$ of t (with respect to the standard basis) as a diagonal matrix (with respect to this eigenvector basis) via the relation $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$. Furthermore, when t has a symmetric matrix representation, the eigenvectors corresponding to different eigenvalues are orthogonal (Theorem C64), and an eigenvector basis can always be found. In addition, the basis vectors can be chosen to give an orthonormal eigenvector basis so that the transition matrix is an orthogonal matrix satisfying $\mathbf{P}^T = \mathbf{P}^{-1}$, giving $\mathbf{D} = \mathbf{P}^T\mathbf{AP}$.

Thus diagonalising matrices involves the main ideas you have studied throughout this book on linear algebra: vectors, matrices, vector spaces, bases and linear transformations.

In the final section you have seen how these techniques can be used to identify the type of a conic, or quadric, from its equation.

Learning outcomes

After working through this unit, you should be able to:

- explain the meaning of the terms *eigenvalue*, *eigenvector*, *characteristic equation* and *eigenspace*
- recognise the geometric interpretation of eigenvectors and eigenspaces in special cases
- find the eigenvalues and eigenvectors of a given 2×2 or 3×3 matrix
- describe some basic properties of eigenvalues and eigenvectors
- write down the matrix of a linear transformation t with respect to a given eigenvector basis of t
- write down the *transition matrix* from an eigenvector basis to the standard basis
- *diagonalise* a given square matrix, if possible
- understand that any symmetric matrix can be *orthogonally diagonalised*
- orthogonally diagonalise a given symmetric matrix
- describe some basic properties of *orthogonal matrices*
- write the equation of a given non-degenerate conic in standard form and hence classify it
- understand the term *quadric* and recognise the six types of quadric covered
- write the equation of a given quadric in standard form and hence classify it.

Solutions to exercises

Solution to Exercise C115

We have

$$t(2,-2) = (2-8, 2+4) = (-6,6) \\ = -3(2,-2)$$

and

$$t(-7,7) = (-7+28, -7-14) = (21,-21) \\ = -3(-7,7).$$

In each case the original vector is scaled by the factor -3.

Solution to Exercise C116

(a) We have $t(0,1) = (4,-2)$, $t(1,2) = (9,-3)$ and $t(4,1) = (8,2)$.

(b) The linear transformation t maps the line joining the points $(0,0)$ and $(4,1)$ to the line joining the points $(0,0)$ and $(8,2)$. But $(8,2) = 2(4,1)$, so these lines are the same and both can be written as $x = 4y$. Therefore the line $x = 4y$ is mapped to itself by the linear transformation t.

(c) We have

$$t(4k,k) = (4k+4k, 4k-2k) = (8k,2k) \\ = 2(4k,k),$$

so any vector lying along the line $x = 4y$ is scaled by the factor 2.

Solution to Exercise C117

(a) A reflection t in the line $y = x$ maps the point (x,y) to the point (y,x). Each point on the line $y = x$ is mapped to itself, since

$$t(k,k) = (k,k) = 1(k,k),$$

so the non-zero vectors (k,k) are eigenvectors with corresponding eigenvalue 1.

Each point on the line $y = -x$ is mapped to another point on the line $y = -x$, since

$$t(k,-k) = (-k,k) = -1(k,-k),$$

so the non-zero vectors $(k,-k)$ are eigenvectors with corresponding eigenvalue -1.

(b) A 2-dilation t maps the point (x,y) to the point $(2x,2y)$. Every line through the origin is mapped to itself; that is, every non-zero vector in the plane is an eigenvector of t. Let k and l be real numbers which are not both zero. Then

$$t(k,l) = (2k,2l) = 2(k,l),$$

so the non-zero vectors (k,l) are eigenvectors with corresponding eigenvalue 2.

(c) An anticlockwise rotation t through $\pi/2$ maps the point (x,y) to the point $(-y,x)$. No line through the origin is mapped to itself by t, so t has no eigenvectors.

(d) An anticlockwise rotation t through π maps the point (x,y) to the point $(-x,-y)$. Each line through the origin is mapped to itself; that is, each non-zero vector in the plane is an eigenvector of t. Let k and l be real numbers that are not both zero. Then

$$t(k,l) = (-k,-l) = -1(k,l),$$

so the non-zero vectors (k,l) are eigenvectors with corresponding eigenvalue -1.

Solution to Exercise C118

(a) We wish to find those vectors (x,y) that are mapped to scalar multiples of themselves; that is, the vectors that satisfy

$$(-5x+3y, 6x-2y) = (\lambda x, \lambda y).$$

Equating coordinates, we obtain the system

$$\begin{aligned} -5x + 3y &= \lambda x \\ 6x - 2y &= \lambda y, \end{aligned}$$

which we write as

$$\begin{aligned} (-5-\lambda)x + \quad\quad 3y &= 0 \\ 6x + (-2-\lambda)y &= 0. \end{aligned}$$

(b) Non-zero solutions to the eigenvector equations exist if and only if the determinant of the coefficient matrix is 0; that is, if and only if

$$\begin{vmatrix} -5-\lambda & 3 \\ 6 & -2-\lambda \end{vmatrix} = 0.$$

We expand the determinant and obtain

$$(-5-\lambda)(-2-\lambda)-18=0,$$

which simplifies to

$$\lambda^2+7\lambda-8=0.$$

The eigenvalues of t are the solutions to this characteristic equation. We have

$$\lambda^2+7\lambda-8=(\lambda-1)(\lambda+8)=0,$$

so the eigenvalues are $\lambda=1$ and $\lambda=-8$.

(c) To find the corresponding eigenvectors, we consider each value of λ in turn.

$\boxed{\lambda=1}$ The eigenvector equations become

$$\begin{aligned}-6x+3y&=0\\ 6x-3y&=0.\end{aligned}$$

These equations are equivalent to the single equation

$$2x-y=0.$$

Thus the eigenvectors corresponding to $\lambda=1$ are the non-zero vectors (x,y) for which $y=2x$; that is, the vectors of the form

$$(k,2k), \quad \text{where } k\neq 0.$$

$\boxed{\lambda=-8}$ The eigenvector equations become

$$\begin{aligned}3x+3y&=0\\ 6x+6y&=0.\end{aligned}$$

These equations are equivalent to the single equation

$$x+y=0.$$

Thus the eigenvectors corresponding to $\lambda=-8$ are the non-zero vectors (x,y) for which $y=-x$; that is, the vectors of the form

$$(k,-k), \quad \text{where } k\neq 0.$$

Thus the eigenvectors of t are the non-zero vectors of the following forms:

$$(k,2k), \text{ corresponding to } \lambda=1,$$
$$(k,-k), \text{ corresponding to } \lambda=-8.$$

Solution to Exercise C119

(a) The matrix of t with respect to the standard basis for $\mathbb{R}^2$ is

$$\mathbf{A}=\begin{pmatrix}1 & 3\\ 2 & -4\end{pmatrix}.$$

We use Strategy C18 to find the eigenvalues and eigenvectors of $\mathbf{A}$, which are the same as those of t.

First we find the eigenvalues of $\mathbf{A}$.

The characteristic equation of $\mathbf{A}$ is $\det(\mathbf{A}-\lambda\mathbf{I})=0$; that is,

$$\begin{vmatrix}1-\lambda & 3\\ 2 & -4-\lambda\end{vmatrix}=0.$$

We expand the determinant and obtain

$$(1-\lambda)(-4-\lambda)-6=0,$$

which simplifies to

$$\lambda^2+3\lambda-10=(\lambda-2)(\lambda+5)=0.$$

The eigenvalues of $\mathbf{A}$ are therefore $\lambda=2$ and $\lambda=-5$.

Next we find the eigenvectors of $\mathbf{A}$.

The eigenvector equations are

$$\begin{aligned}(1-\lambda)x+\qquad 3y&=0\\ 2x+(-4-\lambda)y&=0.\end{aligned}$$

$\boxed{\lambda=2}$ The eigenvector equations become

$$\begin{aligned}-x+3y&=0\\ 2x-6y&=0.\end{aligned}$$

These equations are equivalent to the single equation

$$x-3y=0.$$

Thus the eigenvectors corresponding to $\lambda=2$ are the non-zero vectors for which $x=3y$; that is, the vectors of the form

$$(3k,k), \quad \text{where } k\neq 0.$$

$\boxed{\lambda=-5}$ The eigenvector equations become

$$\begin{aligned}6x+3y&=0\\ 2x+\ y&=0.\end{aligned}$$

These equations are equivalent to the single equation

$$2x+y=0.$$

Thus the eigenvectors corresponding to $\lambda=-5$ are the non-zero vectors for which $y=-2x$; that is, the vectors of the form

$$(k,-2k), \quad \text{where } k\neq 0.$$

Thus the eigenvectors of t are the non-zero vectors of the following forms:

$$(3k, k), \text{ corresponding to } \lambda = 2,$$
$$(k, -2k), \text{ corresponding to } \lambda = -5.$$

(b) The matrix of t with respect to the standard basis for $\mathbb{R}^2$ is

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -2 & -2 \end{pmatrix}.$$

We use Strategy C18 to find the eigenvalues and eigenvectors of $\mathbf{A}$, which are the same as those of t.

First we find the eigenvalues of $\mathbf{A}$.

The characteristic equation of $\mathbf{A}$ is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$; that is,

$$\begin{vmatrix} 1-\lambda & -2 \\ -2 & -2-\lambda \end{vmatrix} = 0.$$

We expand the determinant and obtain

$$(1-\lambda)(-2-\lambda) - 4 = 0,$$

which simplifies to

$$\lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3) = 0.$$

The eigenvalues of $\mathbf{A}$ are therefore $\lambda = 2$ and $\lambda = -3$.

Next we find the eigenvectors of $\mathbf{A}$.

The eigenvector equations are

$$\begin{aligned} (1-\lambda)x - \qquad 2y &= 0 \\ -2x + (-2-\lambda)y &= 0. \end{aligned}$$

$\boxed{\lambda = 2}$ The eigenvector equations become

$$\begin{aligned} -x - 2y &= 0 \\ -2x - 4y &= 0. \end{aligned}$$

These equations are equivalent to the single equation

$$x + 2y = 0.$$

Thus the eigenvectors corresponding to $\lambda = 2$ are the non-zero vectors for which $x = -2y$; that is, the vectors of the form

$$(-2k, k), \quad \text{where } k \neq 0.$$

$\boxed{\lambda = -3}$ The eigenvector equations become

$$\begin{aligned} 4x - 2y &= 0 \\ -2x + y &= 0. \end{aligned}$$

These equations are equivalent to the single equation

$$2x - y = 0.$$

Thus the eigenvectors corresponding to $\lambda = -3$ are the non-zero vectors for which $y = 2x$; that is, the vectors of the form

$$(k, 2k), \quad \text{where } k \neq 0.$$

Thus the eigenvectors of t are the non-zero vectors of the following forms:

$$(-2k, k), \text{ corresponding to } \lambda = 2,$$
$$(k, 2k), \text{ corresponding to } \lambda = -3.$$

Solution to Exercise C120

The matrix of t with respect to the standard basis for $\mathbb{R}^3$ is

$$\mathbf{A} = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 2 \end{pmatrix}.$$

We use Strategy C18 to find the eigenvalues and eigenvectors of $\mathbf{A}$, which are the same as those of t.

First we find the eigenvalues of $\mathbf{A}$.

The characteristic equation is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$; that is,

$$\begin{vmatrix} 4-\lambda & 2 & 0 \\ 2 & 3-\lambda & 2 \\ 0 & 2 & 2-\lambda \end{vmatrix} = 0.$$

We expand the determinant and obtain

$$(4-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 0 & 2-\lambda \end{vmatrix} + 0 = 0.$$

Simplifying this expression, we obtain

$$(4-\lambda)((3-\lambda)(2-\lambda) - 4) - 2(2(2-\lambda)) = 0,$$

or

$$\lambda^3 - 9\lambda^2 + 18\lambda = 0.$$

There is no constant term, so we take out the factor λ, then factorise the remaining quadratic factor:

$$\lambda(\lambda^2 - 9\lambda + 18) = \lambda(\lambda - 6)(\lambda - 3) = 0.$$

The eigenvalues of $\mathbf{A}$ are therefore $\lambda = 0$, $\lambda = 6$ and $\lambda = 3$.

(As a quick check $4+3+2=9=6+3+0$, so the sum of the eigenvalues is indeed equal to the sum of the diagonal entries.)

Next we find the eigenvectors of $\mathbf{A}$.

The eigenvector equations are

$$\begin{aligned}(4-\lambda)x + \quad 2y \qquad\qquad &= 0\\ 2x + (3-\lambda)y + \quad 2z &= 0\\ 2y + (2-\lambda)z &= 0.\end{aligned}$$

$\boxed{\lambda = 6}$ The eigenvector equations become

$$\begin{aligned}-2x + 2y \qquad &= 0\\ 2x - 3y + 2z &= 0\\ 2y - 4z &= 0.\end{aligned}$$

The first and third equations imply that $x = y$ and $y = 2z$, so $x = 2z$. These satisfy the second equation. Thus the eigenvectors corresponding to the eigenvalue $\lambda = 6$ are the non-zero vectors (x, y, z) satisfying $y = 2z$ and $x = 2z$; that is, the vectors of the form

$$(2k, 2k, k), \quad \text{where } k \neq 0.$$

$\boxed{\lambda = 3}$ The eigenvector equations become

$$\begin{aligned}x + 2y \qquad &= 0\\ 2x \qquad + 2z &= 0\\ 2y - \quad z &= 0.\end{aligned}$$

The first and second equations imply that $x = -2y$ and $z = -x$, so $z = 2y$. These satisfy the third equation. Thus the eigenvectors corresponding to the eigenvalue $\lambda = 3$ are the non-zero vectors (x, y, z) satisfying $x = -2y$ and $z = 2y$; that is, the vectors of the form

$$(-2k, k, 2k), \quad \text{where } k \neq 0.$$

$\boxed{\lambda = 0}$ The eigenvector equations become

$$\begin{aligned}4x + 2y \qquad &= 0\\ 2x + 3y + 2z &= 0\\ 2y + 2z &= 0.\end{aligned}$$

The first and third equations imply that $y = -2x$ and $z = -y$, so $z = 2x$. These satisfy the second equation. Thus the eigenvectors corresponding to the eigenvalue $\lambda = 0$ are the non-zero vectors (x, y, z) satisfying $y = -2x$ and $z = 2x$; that is, the vectors of the form

$$(k, -2k, 2k), \quad \text{where } k \neq 0.$$

Thus the eigenvectors of t are the non-zero vectors of the following forms:

$$\begin{aligned}&(2k, 2k, k), \text{ corresponding to } \lambda = 6,\\ &(-2k, k, 2k), \text{ corresponding to } \lambda = 3,\\ &(k, -2k, 2k), \text{ corresponding to } \lambda = 0.\end{aligned}$$

Solution to Exercise C121

(a) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 6 \end{pmatrix}.$$

The characteristic equation is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$; that is,

$$\begin{vmatrix} 1-\lambda & 2 \\ 0 & 6-\lambda \end{vmatrix} = 0.$$

We expand the determinant and obtain

$$(1-\lambda)(6-\lambda) - 0 = 0.$$

The eigenvalues of $\mathbf{A}$ are therefore $\lambda = 1$ and $\lambda = 6$. Notice that these are the diagonal entries of the upper triangular matrix $\mathbf{A}$.

(b) Let

$$\mathbf{A} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 21 \end{pmatrix}.$$

The characteristic equation is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$; that is,

$$\begin{vmatrix} 8-\lambda & 0 & 0 \\ 0 & -5-\lambda & 0 \\ 0 & 0 & 21-\lambda \end{vmatrix} = 0.$$

We expand the determinant and obtain

$$(8-\lambda)\begin{vmatrix} -5-\lambda & 0 \\ 0 & 21-\lambda \end{vmatrix} = 0.$$

Simplifying this expression, we obtain

$$(8-\lambda)((-5-\lambda)(21-\lambda) - 0) = 0.$$

The eigenvalues of $\mathbf{A}$ are therefore $\lambda = 8$, $\lambda = -5$ and $\lambda = 21$. Again, these are the diagonal entries of the diagonal matrix $\mathbf{A}$.

(c) Let

$$\mathbf{A} = \begin{pmatrix} 4 & 0 & 0 \\ 25 & -2 & 0 \\ 17 & \pi & 6 \end{pmatrix}.$$

The characteristic equation is $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$; that is,

$$\begin{vmatrix} 4-\lambda & 0 & 0 \\ 25 & -2-\lambda & 0 \\ 17 & \pi & 6-\lambda \end{vmatrix} = 0.$$

We expand the determinant and obtain

$$(4-\lambda)\begin{vmatrix} -2-\lambda & 0 \\ \pi & 6-\lambda \end{vmatrix} = 0.$$

Simplifying this expression, we obtain

$$(4-\lambda)((-2-\lambda)(6-\lambda) - 0) = 0.$$

The eigenvalues of $\mathbf{A}$ are therefore $\lambda = 4$, $\lambda = -2$ and $\lambda = 6$. Again, these are the diagonal entries of the lower triangular matrix $\mathbf{A}$.

Solution to Exercise C122

$\boxed{\lambda = 6}$ The non-zero vectors of the form $(2k, 2k, k)$ are the eigenvectors of t corresponding to $\lambda = 6$. The eigenspace $S(6)$ is therefore the set of vectors

$$\{(2k, 2k, k) : k \in \mathbb{R}\}.$$

Any vector in $S(6)$ can be written as $k(2, 2, 1)$, so $\{(2, 2, 1)\}$ is a basis for $S(6)$.

Thus $S(6)$ has dimension 1.

$\boxed{\lambda = 3}$ The non-zero vectors of the form $(-2k, k, 2k)$ are the eigenvectors of t corresponding to $\lambda = 3$. The eigenspace $S(3)$ is therefore the set of vectors

$$\{(-2k, k, 2k) : k \in \mathbb{R}\}.$$

Any vector in $S(3)$ can be written as $k(-2, 1, 2)$, so $\{(-2, 1, 2)\}$ is a basis for $S(3)$.

Thus $S(3)$ has dimension 1.

Solution to Exercise C123

The matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

is triangular, so the eigenvalues are the diagonal entries $\lambda = 1$, $\lambda = 4$ and $\lambda = 4$.

The eigenvector equations are

$$\begin{aligned} (1-\lambda)x + \quad y - \quad z &= 0 \\ (4-\lambda)y \quad\quad &= 0 \\ (4-\lambda)z &= 0. \end{aligned}$$

$\boxed{\lambda = 1}$ The eigenvalue $\lambda = 1$ has multiplicity 1.

The eigenvector equations become

$$\begin{aligned} y - \quad z &= 0 \\ 3y \quad\quad &= 0 \\ 3z &= 0. \end{aligned}$$

The second and third equations give $y = 0$ and $z = 0$, respectively, which satisfy the first equation. (They give no constraint on x.)

Thus the eigenvectors corresponding to the eigenvalue $\lambda = 1$ are the vectors of the form $(k, 0, 0)$, where $k \neq 0$.

The eigenspace $S(1)$ is the set of vectors

$$\{(k, 0, 0) : k \in \mathbb{R}\}.$$

Any vector in $S(1)$ can be written as $k(1, 0, 0)$, so

$$\{(1, 0, 0)\}$$

is a basis for $S(1)$.

Thus $S(1)$ has dimension 1.

(Geometrically, $S(1)$ is the x-axis.)

$\boxed{\lambda = 4}$ The eigenvalue $\lambda = 4$ has multiplicity 2.

The eigenvector equations become

$$\begin{aligned} -3x + \quad y - \quad z &= 0 \\ 0y \quad\quad &= 0 \\ 0z &= 0. \end{aligned}$$

The first equation gives $z = y - 3x$ and the second and third give no constraints on y and z.

Thus the eigenvectors corresponding to the eigenvalue $\lambda = 4$ are the vectors of the form $(k, l, l - 3k)$, where k and l are not both 0.

The eigenspace $S(4)$ is the set of vectors

$$\{(k, l, l - 3k) : k, l \in \mathbb{R}\}.$$

Any vector in $S(4)$ can be written as $k(1,0,-3)+l(0,1,1)$, so

$$\{(1,0,-3),(0,1,1)\}$$

is a basis for $S(4)$.

Thus $S(4)$ has dimension 2.

(Geometrically, $S(4)$ is the plane in $\mathbb{R}^3$ $-3x+y-z=0$.)

An alternative solution comes from using the equivalent equation $x=\frac{1}{3}(y-z)$, and has basis

$$\left\{\left(\tfrac{1}{3},1,0\right),\left(-\tfrac{1}{3},0,1\right)\right\}.$$

Solution to Exercise C124

The matrix

$$\mathbf{A}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

is triangular, so the eigenvalues are the diagonal entries $\lambda=1$ and $\lambda=1$.

The eigenvector equations are

$$\begin{aligned}(1-\lambda)x+\quad\quad y&=0\\ (1-\lambda)y&=0.\end{aligned}$$

$\boxed{\lambda=1}$ The eigenvalue $\lambda=1$ has multiplicity 2.

The eigenvector equations become

$$\begin{aligned}0x+\;\;y&=0\\ 0y&=0.\end{aligned}$$

Thus $y=0$ and there are no constraints on x. Thus the eigenvectors corresponding to the eigenvalue $\lambda=1$ are the vectors of the form $(k,0)$, where $k\neq 0$.

The eigenspace $S(1)$ is the set of vectors

$$\{(k,0):k\in\mathbb{R}\}.$$

Any vector in $S(1)$ can be written as $k(1,0)$, so

$$\{(1,0)\}$$

is a basis for $S(1)$.

Thus $S(1)$ has dimension 1.

(Geometrically, $S(1)$ is the x-axis in $\mathbb{R}^2$.)

Solution to Exercise C125

Let

$$\mathbf{A}=\begin{pmatrix}1 & -1 & 0\\ 1 & 4 & 1\\ -1 & 1 & 4\end{pmatrix}.$$

The characteristic equation is $\det(\mathbf{A}-\lambda\mathbf{I})=0$; that is,

$$\begin{vmatrix}1-\lambda & -1 & 0\\ 1 & 4-\lambda & 1\\ -1 & 1 & 4-\lambda\end{vmatrix}=0.$$

We expand the determinant and obtain

$$(1-\lambda)\begin{vmatrix}4-\lambda & 1\\ 1 & 4-\lambda\end{vmatrix}+\begin{vmatrix}1 & 1\\ -1 & 4-\lambda\end{vmatrix}=0.$$

This simplifies to

$$(1-\lambda)((4-\lambda)^2-1)+((4-\lambda)+1)=0.$$

Using the relation $x^2-1=(x-1)(x+1)$, where $x=4-\lambda$, this simplifies further to

$$(1-\lambda)(3-\lambda)(5-\lambda)+(5-\lambda)=0,$$

and thus

$$\begin{aligned}(5-\lambda)((1-\lambda)(3-\lambda)+1)&=(5-\lambda)(\lambda^2-4\lambda+4)\\ &=(5-\lambda)(\lambda-2)^2\\ &=0.\end{aligned}$$

The eigenvalues of $\mathbf{A}$ are $\lambda=5$, $\lambda=2$ and $\lambda=2$.

(As a quick check $1+4+4=9=5+2+2$, so the sum of the eigenvalues is indeed equal to the sum of the diagonal entries.)

The eigenvector equations are

$$\begin{aligned}(1-\lambda)x-\quad\quad y\quad\quad\quad\quad&=0\\ x+(4-\lambda)y+\quad\quad z&=0\\ -x+\quad\quad y+(4-\lambda)z&=0.\end{aligned}$$

$\boxed{\lambda=5}$ The eigenvalue $\lambda=5$ has multiplicity 1.

The eigenvector equations become

$$\begin{aligned}-4x-y\quad\quad&=0\\ x-y+z&=0\\ -x+y-z&=0.\end{aligned}$$

The first equation gives $y=-4x$ and substituting this into the second gives $5x+z=0$, which implies that $z=-5x$. The third equation is equivalent to the second.

Thus the eigenvectors corresponding to the eigenvalue $\lambda = 5$ are the vectors of the form $(k, -4k, -5k)$, where $k \neq 0$.

The eigenspace $S(5)$ is the set of vectors

$$\{(k, -4k, -5k) : k \in \mathbb{R}\}.$$

Any vector in $S(5)$ can be written as $k(1, -4, -5)$, so

$$\{(1, -4, -5)\}$$

is a basis for $S(5)$.

Thus $S(5)$ has dimension 1.

$\boxed{\lambda = 2}$ The eigenvalue $\lambda = 2$ has multiplicity 2.

The eigenvector equations become

$$\begin{aligned} -x - y \phantom{{}+2z} &= 0 \\ x + 2y + z &= 0 \\ -x + y + 2z &= 0. \end{aligned}$$

The first equation gives $y = -x$ and substituting this into the second gives $-x + z = 0$, which implies that $z = x$. These satisfy the third equation.

Thus the eigenvectors corresponding to the eigenvalue $\lambda = 2$ are the vectors of the form $(k, -k, k)$, where $k \neq 0$.

The eigenspace $S(2)$ is the set of vectors

$$\{(k, -k, k) : k \in \mathbb{R}\}.$$

Any vector in $S(2)$ can be written as $k(1, -1, 1)$, so

$$\{(1, -1, 1)\}$$

is a basis for $S(2)$.

Thus $S(2)$ has dimension 1.

Solution to Exercise C126

Letting $k = 1$, we see that $(-2, 1)$ and $(1, 2)$ are eigenvectors of t. Since $(1, 2)$ is not a multiple of $(-2, 1)$, these two eigenvectors form a basis for $\mathbb{R}^2$.

Solution to Exercise C127

Each of the vectors in E is an eigenvector of t:

$$\begin{aligned} t(0, 1, -1) &= (0, 0, 0) = 0(0, 1, -1), \\ t(-2, 1, 0) &= (4, -2, 0) = -2(-2, 1, 0), \\ t(1, 0, -1) &= (-3, 0, 3) = -3(1, 0, -1). \end{aligned}$$

Thus E is a basis for $\mathbb{R}^3$ consisting of eigenvectors of t; that is, E is an eigenvector basis of t.

Solution to Exercise C128

(a) The matrix of t with respect to the standard basis for $\mathbb{R}^2$ is

$$\begin{pmatrix} 1 & -2 \\ -2 & -2 \end{pmatrix}.$$

(b) Following Strategy C19, first we find the images of the vectors in the basis $E = \{(-2, 1), (1, 2)\}$:

$$t(-2, 1) = (-4, 2), \quad t(1, 2) = (-3, -6).$$

Next we find the E-coordinates of each of these image vectors:

$$\begin{aligned} (-4, 2) &= 2(-2, 1) + 0(1, 2) \\ &= (2, 0)_E, \\ (-3, -6) &= 0(-2, 1) - 3(1, 2) \\ &= (0, -3)_E. \end{aligned}$$

Therefore $t(-2, 1) = (2, 0)_E$ and $t(1, 2) = (0, -3)_E$. So the matrix of t with respect to the eigenvector basis E is

$$\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}.$$

Solution to Exercise C129

In Exercise C127 you showed that

$$\begin{aligned} t(0, 1, -1) &= 0(0, 1, -1), \\ t(-2, 1, 0) &= -2(-2, 1, 0), \\ t(1, 0, -1) &= -3(1, 0, -1). \end{aligned}$$

So the eigenvalues of t are $\lambda_1 = 0$, $\lambda_2 = -2$ and $\lambda_3 = -3$, and, by Theorem C59, the matrix of t with respect to E is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

Solution to Exercise C130

(a) $\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$

(b) $\mathbf{P} = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$

Solution to Exercise C131

Let $t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation given by

$$t(x, y) = (x - 2y, -2x - 2y)$$

and let E be the eigenvector basis $\{(-2, 1), (1, 2)\}$ of t. It follows from Exercise C128 that $\mathbf{A}$ is the matrix of t with respect to the standard basis for $\mathbb{R}^2$ and $\mathbf{D}$ is the matrix of t with respect to the eigenvector basis E. By Theorem C62, $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$, where $\mathbf{P}$ is the transition matrix from E to the standard basis for $\mathbb{R}^2$; that is,

$$\mathbf{P} = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Solution to Exercise C132

(a) $\mathbf{D}^5 = \begin{pmatrix} 2^5 & 0 \\ 0 & (-3)^5 \end{pmatrix} = \begin{pmatrix} 32 & 0 \\ 0 & -243 \end{pmatrix}$

(b) We have $\mathbf{A}^5 = \mathbf{PD}^5\mathbf{P}^{-1}$, where $\mathbf{D}$ is as in part (a) and

$$\mathbf{P} = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Since $\mathbf{P}^{-1} = -\frac{1}{5}\begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix}$, it follows that

$$\begin{aligned}\mathbf{A}^5 &= \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 32 & 0 \\ 0 & -243 \end{pmatrix}\begin{pmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \\ &= \begin{pmatrix} -23 & -110 \\ -110 & -188 \end{pmatrix}.\end{aligned}$$

Solution to Exercise C133

(There are many solutions possible for this and for each of the remaining exercises in this section, each corresponding to a different ordering of the eigenvalues or a different choice of eigenvectors; in each case the matrix $\mathbf{P}$ should correspond to the matrix $\mathbf{D}$ so that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$.)

We use Strategy C20.

The eigenvalues of $\mathbf{A}$ are $\lambda = 6$, $\lambda = 3$ and $\lambda = 0$.

The eigenvectors of $\mathbf{A}$ are the non-zero vectors of the following forms:

$$\begin{aligned}&(2k, 2k, k), \text{ corresponding to } \lambda = 6,\\ &(-2k, k, 2k), \text{ corresponding to } \lambda = 3,\\ &(k, -2k, 2k), \text{ corresponding to } \lambda = 0.\end{aligned}$$

It follows from Theorem C63 that we can form an eigenvector basis of $\mathbf{A}$ by taking one eigenvector corresponding to each of the three distinct eigenvalues. For example,

$$E = \{(2, 2, 1), (-2, 1, 2), (1, -2, 2)\}$$

is an eigenvector basis of $\mathbf{A}$.

We use the eigenvectors in E to form the columns of the transition matrix:

$$\mathbf{P} = \begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix}.$$

We use the eigenvalues corresponding to the eigenvectors in E to form the diagonal matrix:

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solution to Exercise C134

The characteristic equation of $\mathbf{A}$ is

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0.$$

We expand the determinant and obtain

$$(1-\lambda)((2-\lambda)^2 - 1) = 0,$$

which simplifies to

$$\begin{aligned}(1-\lambda)(\lambda^2 - 4\lambda + 3) &= (1-\lambda)(\lambda-1)(\lambda-3)\\ &= 0.\end{aligned}$$

The eigenvalues of $\mathbf{A}$ are therefore $\lambda = 3$, $\lambda = 1$ and $\lambda = 1$.

To find the eigenspaces of $\mathbf{A}$, we consider the eigenvector equations

$$\begin{aligned}(1-\lambda)x \qquad\qquad\qquad\qquad &= 0\\ (2-\lambda)y + \qquad z &= 0\\ y + (2-\lambda)z &= 0,\end{aligned}$$

for each of the eigenvalues.

$\boxed{\lambda = 3}$ The eigenvector equations become

$$\begin{aligned} -2x \qquad\quad &= 0 \\ -y + z &= 0 \\ y - z &= 0. \end{aligned}$$

So $x = 0$, $y = z$.

Thus $S(3) = \{(0, k, k) : k \in \mathbb{R}\}$.

$\boxed{\lambda = 1}$ The eigenvector equations become

$$\begin{aligned} 0x \qquad &= 0 \\ y + z &= 0 \\ y + z &= 0. \end{aligned}$$

So $z = -y$ and there are no constraints on x.

Thus $S(1) = \{(k, l, -l) : k, l \in \mathbb{R}\}$.

A basis for $S(3)$ is $\{(0, 1, 1)\}$ and a basis for $S(1)$ is $\{(1, 0, 0), (0, 1, -1)\}$ because any vector in $S(1)$ can be written as $k(1, 0, 0) + l(0, 1, -1)$. The set

$$E = \{(0, 1, 1), (1, 0, 0), (0, 1, -1)\}$$

contains three vectors, so it is an eigenvector basis of $\mathbf{A}$.

We use the eigenvectors in E to form the columns of the transition matrix:

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

We use the eigenvalues corresponding to the eigenvectors in E to form the diagonal matrix:

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution to Exercise C135

(a) We have

$$\begin{aligned} (2k, 2k, k) \cdot (-2l, l, 2l) &= -4kl + 2kl + 2kl \\ &= 0, \\ (2k, 2k, k) \cdot (m, -2m, 2m) &= 2km - 4km + 2km \\ &= 0, \\ (-2l, l, 2l) \cdot (m, -2m, 2m) &= -2lm - 2lm + 4lm \\ &= 0. \end{aligned}$$

Thus the given vectors form an orthogonal set. Since there are three of them, they form an orthogonal basis for $\mathbb{R}^3$.

(b)

$$\begin{aligned} |\mathbf{v}_1| = |(2k, 2k, k)| &= \sqrt{4k^2 + 4k^2 + k^2} \\ &= \sqrt{9k^2} \\ &= 3k, \\ |\mathbf{v}_2| = |(-2l, l, 2l)| &= \sqrt{4l^2 + l^2 + 4l^2} \\ &= \sqrt{9l^2} \\ &= 3l, \\ |\mathbf{v}_3| = |(m, -2m, 2m)| &= \sqrt{m^2 + 4m^2 + 4m^2} \\ &= \sqrt{9m^2} \\ &= 3m. \end{aligned}$$

Thus $|\mathbf{v}_1| = |\mathbf{v}_2| = |\mathbf{v}_3| = 1$ if

$$k = l = m = \tfrac{1}{3}.$$

Solution to Exercise C136

We calculate $\mathbf{P}^T\mathbf{P}$.

$$\begin{aligned} \mathbf{P}^T\mathbf{P} &= \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{9}{9} & 0 & 0 \\ 0 & \frac{9}{9} & 0 \\ 0 & 0 & \frac{9}{9} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \end{aligned}$$

Solution to Exercise C137

(a) We use Strategy C22.

The characteristic equation of $\mathbf{A}$ is

$$\begin{vmatrix} 9 - \lambda & -2 \\ -2 & 6 - \lambda \end{vmatrix} = 0.$$

We expand the determinant and obtain

$$(9 - \lambda)(6 - \lambda) - 4 = 0,$$

which simplifies to

$$\lambda^2 - 15\lambda + 50 = (\lambda - 10)(\lambda - 5) = 0.$$

The eigenvalues of $\mathbf{A}$ are therefore $\lambda = 10$ and $\lambda = 5$.

Next we find orthonormal bases for the eigenspaces.

The eigenvector equations are

$$\begin{aligned} (9 - \lambda)x - \qquad 2y &= 0 \\ -2x + (6 - \lambda)y &= 0. \end{aligned}$$

$\boxed{\lambda = 10}$ The eigenvector equations become

$$\begin{aligned} -x - 2y &= 0 \\ -2x - 4y &= 0. \end{aligned}$$

These equations are equivalent to the single equation

$$x + 2y = 0,$$

that is, $x = -2y$. Thus the eigenvectors corresponding to $\lambda = 10$ are the non-zero vectors of the form $(-2k, k)$.

An eigenvector of magnitude 1 corresponding to $\lambda = 10$ is

$$\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right).$$

$\boxed{\lambda = 5}$ The eigenvector equations become

$$\begin{aligned} 4x - 2y &= 0 \\ -2x + y &= 0. \end{aligned}$$

These equations are equivalent to the single equation

$$2x - y = 0,$$

that is, $y = 2x$. Thus the eigenvectors corresponding to $\lambda = 5$ are the non-zero vectors of the form $(k, 2k)$.

An eigenvector of magnitude 1 corresponding to $\lambda = 5$ is

$$\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right).$$

It follows from Theorem C64 that an orthonormal eigenvector basis of $\mathbf{A}$ is

$$E = \left\{\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)\right\}.$$

We use the eigenvectors in E to form the columns of the transition matrix:

$$\mathbf{P} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}.$$

We use the eigenvalues corresponding to the eigenvectors in E to form the diagonal matrix:

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}.$$

(b) The eigenvalues of $\mathbf{A}$ are given as $\lambda = 6$, $\lambda = 3$ and $\lambda = 2$.

Now we find an orthonormal eigenvector basis of $\mathbf{A}$.

The eigenvector equations are

$$\begin{aligned} (5-\lambda)x - y - z &= 0 \\ -x + (3-\lambda)y + z &= 0 \\ -x + y + (3-\lambda)z &= 0. \end{aligned}$$

$\boxed{\lambda = 6}$ The eigenvector equations become

$$\begin{aligned} -x - y - z &= 0 \\ -x - 3y + z &= 0 \\ -x + y - 3z &= 0. \end{aligned}$$

Adding the first and second equations together, we obtain

$$-2x - 4y = 0,$$

so $x = -2y$. Substituting this into the third equation, we obtain

$$3y - 3z = 0,$$

so $z = y$. Thus the eigenvectors corresponding to $\lambda = 6$ are the non-zero vectors of the form $(-2k, k, k)$.

An eigenvector of magnitude 1 corresponding to $\lambda = 6$ is

$$\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right).$$

$\boxed{\lambda = 3}$ The eigenvector equations become

$$\begin{aligned} 2x - y - z &= 0 \\ -x \qquad + z &= 0 \\ -x + y \qquad &= 0. \end{aligned}$$

The second and third equations imply that $z = x$ and $y = x$. These satisfy the first equation. Thus the eigenvectors corresponding to $\lambda = 3$ are the non-zero vectors of the form (k, k, k).

An eigenvector of magnitude 1 corresponding to $\lambda = 3$ is

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

$\boxed{\lambda = 2}$ The eigenvector equations become

$$\begin{aligned} 3x - y - z &= 0 \\ -x + y + z &= 0 \\ -x + y + z &= 0. \end{aligned}$$

Adding the first and second equations together, we obtain

$$2x = 0,$$

which implies that $x = 0$. Substituting this into the third equation, we obtain

$$y + z = 0,$$

which implies that $z = -y$. Thus the eigenvectors corresponding to $\lambda = 2$ are the non-zero vectors of the form $(0, k, -k)$.

An eigenvector of magnitude 1 corresponding to $\lambda = 2$ is

$$\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

It follows from Theorem C64 that an orthonormal eigenvector basis of $\mathbf{A}$ is

$$E = \left\{\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\}.$$

We use the eigenvectors in E to form the columns of the transition matrix:

$$\mathbf{P} = \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

We use the eigenvalues corresponding to the eigenvectors in E to form the diagonal matrix:

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Solution to Exercise C138

We use Strategy C22.

A basis for the eigenspace $S(3)$ is $\{(0, 1, 1)\}$, so an orthonormal basis for $S(3)$ is

$$\left\{\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}.$$

A basis for the eigenspace $S(1)$ is $\{(1, 0, 0), (0, 1, -1)\}$.

These two basis vectors are orthogonal since

$$(1, 0, 0) \cdot (0, 1, -1) = 0.$$

An orthonormal basis for $S(1)$ is therefore

$$\left\{(1, 0, 0), \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\}.$$

By Theorem C64 an orthonormal eigenvector basis of $\mathbf{A}$ is therefore

$$E = \left\{\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), (1, 0, 0), \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\}.$$

We use the eigenvectors in E to form the columns of the transition matrix:

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

We use the eigenvalues corresponding to the eigenvectors in E to form the diagonal matrix:

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution to Exercise C139

By Theorem C65, to prove that the product $\mathbf{PQ}$ is orthogonal it is sufficient to show that

$$(\mathbf{PQ})^T = (\mathbf{PQ})^{-1}.$$

But

$$(\mathbf{PQ})^T = \mathbf{Q}^T\mathbf{P}^T = \mathbf{Q}^{-1}\mathbf{P}^{-1} = (\mathbf{PQ})^{-1}.$$

Solution to Exercise C140

(a) To verify that $\mathbf{A}$ is orthogonal, it is sufficient to show that $\mathbf{A}^T\mathbf{A} = \mathbf{I}$, by Theorem C65.

$$\begin{aligned} \mathbf{A}^T\mathbf{A} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}, \end{aligned}$$

so $\mathbf{A}$ is orthogonal.

(Alternatively, we could have shown that the vectors $(0, 0, 1)$, $(0, 1, 0)$ and $(-1, 0, 0)$ form an orthonormal basis for $\mathbb{R}^3$.)

(b) We evaluate the determinant of $\mathbf{A}$:

$$\begin{vmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0 - 0 - \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1.$$

Therefore $\mathbf{A}$ represents a rotation of $\mathbb{R}^3$.

Solution to Exercise C141

(a) The ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is written in matrix form as

$$\mathbf{x}^T \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & 0 \end{pmatrix} \mathbf{x} - 1 = 0.$$

So the ellipse in standard position has

$$\mathbf{A} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \text{ and } \mathbf{J} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(b) The hyperbola with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is written in matrix form as

$$\mathbf{x}^T \begin{pmatrix} 1/a^2 & 0 \\ 0 & -1/b^2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & 0 \end{pmatrix} \mathbf{x} - 1 = 0.$$

So the hyperbola in standard position has

$$\mathbf{A} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & -1/b^2 \end{pmatrix} \text{ and } \mathbf{J} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(c) The parabola with equation

$$y^2 = 4ax$$

is written in matrix form as

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4a & 0 \end{pmatrix} \mathbf{x} + 0 = 0.$$

So the parabola in standard position has

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \mathbf{J} = \begin{pmatrix} -4a \\ 0 \end{pmatrix}.$$

Solution to Exercise C142

By Theorem C64 an orthonormal eigenvector basis of $\mathbf{A}$ for the eigenvalues $\lambda = 2$ and $\lambda = -3$, in that order, is

$$E = \left\{ \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right), \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) \right\}.$$

We use the eigenvectors in E to form the columns of the transition matrix:

$$\mathbf{P} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}.$$

We use the eigenvalues to form the diagonal matrix

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}.$$

It follows from equation (9) that the equation of the conic is now

$$\begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} 6 & 12 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + 21 = 0,$$

that is,

$$2(x')^2 - 3(y')^2 + 6\sqrt{5}y' + 21 = 0.$$

Solution to Exercise C143

We have

$$2(x')^2 - 3(y')^2 + 6\sqrt{5}y' + 21 = 0,$$

which is equivalent to

$$2(x')^2 - 3\left((y')^2 - 2\sqrt{5}y'\right) + 21 = 0.$$

Completing the square gives

$$2(x')^2 - 3(y' - \sqrt{5})^2 + 15 + 21 = 0,$$

so

$$2(x')^2 - 3(y' - \sqrt{5})^2 + 36 = 0.$$

We set the new coordinates to be

$$\mathbf{x}'' = (x'', y'') = (x', y' - \sqrt{5}),$$

so substitute $x'' = x'$ and $y'' = y' - \sqrt{5}$.

The equation of the conic is now

$$2(x'')^2 - 3(y'')^2 = -36,$$

or

$$-\frac{(x'')^2}{18} + \frac{(y'')^2}{12} = 1.$$

Solution to Exercise C144

1. Introduce matrices.

   We have
   $$\mathbf{A} = \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} -10 \\ -20 \end{pmatrix}.$$

2. Align the axes.

   We have
   $$\mathbf{P}^T\mathbf{AP} = \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix},$$
   where
   $$\mathbf{P} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}.$$
   So
   $$\begin{aligned} (f \quad g) &= (-10 \quad -20) \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\frac{50}{\sqrt{5}} \end{pmatrix} \\ &= (0 \quad -10\sqrt{5}). \end{aligned}$$
   The equation of the conic is now
   $$10(x')^2 + 5(y')^2 - 10\sqrt{5}y' - 5 = 0.$$
   Dividing through by 5, we obtain
   $$2(x')^2 + (y')^2 - 2\sqrt{5}y' - 1 = 0.$$

3. Translate the origin.

   We write this equation as
   $$2(x')^2 + \left((y')^2 - 2\sqrt{5}y'\right) - 1 = 0.$$
   Completing the square in this equation, we obtain
   $$2(x')^2 + (y' - \sqrt{5})^2 - 5 - 1 = 0.$$
   Substituting $x'' = x'$ and $y'' = y' - \sqrt{5}$ in this equation and simplifying, we obtain
   $$2(x'')^2 + (y'')^2 - 6 = 0.$$

The equation of the conic in standard form is
$$\frac{(x'')^2}{3} + \frac{(y'')^2}{6} = 1.$$
The conic is an ellipse.

Solution to Exercise C145

1. Introduce matrices.

   We have
   $$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} -6 \\ -8 \end{pmatrix}.$$

2. Align the axes.

   The characteristic equation of $\mathbf{A}$ is
   $$\begin{vmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} = 0.$$
   We expand the determinant and obtain
   $$(1-\lambda)(4-\lambda) - 4 = 0,$$
   which simplifies to
   $$\lambda^2 - 5\lambda = \lambda(\lambda - 5) = 0.$$
   The eigenvalues of $\mathbf{A}$ are 5 and 0.

   The eigenvector equations are
   $$\begin{aligned} (1-\lambda)x - \quad\quad 2y &= 0 \\ -2x + (4-\lambda)y &= 0. \end{aligned}$$

   $\boxed{\lambda = 5}$ The eigenvector equations become
   $$\begin{aligned} -4x - 2y &= 0 \\ -2x - y &= 0. \end{aligned}$$
   These equations are equivalent to the single equation
   $$2x + y = 0,$$
   which implies that $y = -2x$. Thus the eigenvectors corresponding to $\lambda = 5$ are the non-zero vectors of the form $(k, -2k)$.

   An orthonormal basis for $S(5)$ is
   $$\left\{\left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)\right\}.$$

   $\boxed{\lambda = 0}$ The eigenvector equations become
   $$\begin{aligned} x - 2y &= 0 \\ -2x + 4y &= 0. \end{aligned}$$
   These equations are equivalent to the single equation
   $$x - 2y = 0,$$
   which implies that $x = 2y$. Thus the eigenvectors corresponding to $\lambda = 0$ are the non-zero vectors of the form $(2k, k)$.

An orthonormal basis for $S(0)$ is

$$\left\{\left(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}\right)\right\}.$$

By Theorem C64 an orthonormal eigenvector basis of $\mathbf{A}$ is therefore

$$E=\left\{\left(\frac{1}{\sqrt{5}},-\frac{2}{\sqrt{5}}\right),\left(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}\right)\right\}.$$

We use the eigenvectors in E to form the columns of the transition matrix:

$$\mathbf{P}=\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}.$$

Now,

$$\mathbf{P}^T\mathbf{A}\mathbf{P}=\begin{pmatrix}5 & 0\\ 0 & 0\end{pmatrix}$$

so

$$\begin{aligned}(f\quad g)&=(-6\quad -8)\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\\ &=\begin{pmatrix}\frac{10}{\sqrt{5}} & -\frac{20}{\sqrt{5}}\end{pmatrix}\\ &=\begin{pmatrix}2\sqrt{5} & -4\sqrt{5}\end{pmatrix}.\end{aligned}$$

The equation of the conic is now

$$5(x')^2+2\sqrt{5}x'-4\sqrt{5}y'+5=0.$$

3. Translate the origin.

We rewrite this equation by taking out the coefficient of the $(x')^2$ term to get

$$5\left((x')^2+\frac{2}{\sqrt{5}}x'\right)-4\sqrt{5}y'+5=0.$$

Completing the square in this equation, we obtain

$$5\left(x'+\frac{1}{\sqrt{5}}\right)^2-1-4\sqrt{5}y'+5=0.$$

We substitute

$$x''=x'+\frac{1}{\sqrt{5}}$$

into this equation and rewrite it by taking out the coefficient of the y' term to get

$$5(x'')^2-4\sqrt{5}\left(y'-\frac{1}{\sqrt{5}}\right)=0.$$

We substitute

$$y''=y'-\frac{1}{\sqrt{5}}$$

to obtain

$$5(x'')^2-4\sqrt{5}y''=0.$$

The equation of the conic in standard form is

$$(x'')^2=\frac{4}{\sqrt{5}}y''.$$

The conic is a parabola.

Solution to Exercise C146

1. Introduce matrices.

We have

$$\mathbf{A}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\quad \mathbf{J}=\begin{pmatrix}-2\\ 4\\ -6\end{pmatrix}.$$

2. Align the axes.

The matrix is already in diagonal form. (The axes of the quadric are parallel to the x-axis, y-axis and z-axis of $\mathbb{R}^3$.)

3. Translate the origin.

We write the equation as

$$(x^2-2x)+(y^2+4y)+(z^2-6z)-11=0.$$

Completing the squares in this equation, we obtain

$$\begin{aligned}(x-1)^2-1+(y+2)^2-4&\\ +(z-3)^2-9-11&=0.\end{aligned}$$

Substituting

$$x'=x-1,\quad y'=y+2\quad\text{and}\quad z'=z-3$$

in this equation and simplifying, we obtain

$$(x')^2+(y')^2+(z')^2-25=0.$$

The equation of the quadric in standard form is

$$\frac{(x')^2}{25}+\frac{(y')^2}{25}+\frac{(z')^2}{25}=1.$$

This is the equation of an ellipsoid.

(This ellipsoid is in fact a sphere since $a=b=c=5$; all the curves of intersection are circles.)

Solution to Exercise C147

1. Introduce matrices. We have

$$\mathbf{A} = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 2 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 12 \\ 0 \\ 12 \end{pmatrix}.$$

2. Align the axes.

   We have

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

   where

$$\mathbf{P} = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}.$$

   So

$$\begin{aligned} \begin{pmatrix} f & g & h \end{pmatrix} &= \begin{pmatrix} 12 & 0 & 12 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \\ &= \begin{pmatrix} 12 & 0 & 12 \end{pmatrix}. \end{aligned}$$

   The equation of the quadric is now

$$6(x')^2 + 3(y')^2 + 12x' + 12z' + 18 = 0.$$

3. Translate the origin.

   We write this equation as

$$6\left((x')^2 + 2x'\right) + 3(y')^2 + 12z' + 18 = 0.$$

   Completing the square in this equation, we obtain

$$6(x'+1)^2 - 6 + 3(y')^2 + 12z' + 18 = 0.$$

   Substituting

$$x'' = x' + 1, \quad y'' = y' \quad \text{and} \quad z'' = z' + 1$$

   in this equation and simplifying, we obtain

$$2(x'')^2 + (y'')^2 + 4z'' = 0.$$

The equation of the quadric in standard form is

$$\frac{(x'')^2}{2} + \frac{(y'')^2}{4} = -z''.$$

This is the equation of an elliptic paraboloid.

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